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Abstract

We consider anisotropic elliptic and parabolic operators in a bounded stratified media in $\mathbb{R}^n$ characterized by discontinuities of the coefficients in one direction. The surfaces of discontinuities cross the boundary of the domain. We prove Carleman estimates for these operators with an arbitrary observation region.

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Contents

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1 Introduction, notation and main results

Consider a bounded open set $\Omega \subset \mathbb{R}^n$. For a second-order elliptic operator, say $A = -\Delta_x$, Carleman estimates take the form[1]

$$s^3 \|e^{s\varphi} u\|^2_{L^2(\Omega)} + s\|e^{s\varphi} \nabla_x u\|^2_{L^2(\Omega)} \lesssim \|e^{s\varphi} A u\|^2_{L^2(\Omega)}, \quad u \in \mathscr{C}_c^\infty(\Omega), \ \ s \geq s_0,$$

for a properly chosen weight function $\varphi(x)$ and s_0 sufficiently large (see e.g. [17]). It is common to use a weight function of the form $\varphi(x) = e^{\lambda\beta(x)}$, with β such that $|\beta'| \neq 0$ and λ sufficiently large. Including a second large parameter (see [16]), the Carleman estimate then takes the form

$$s^3\lambda^4 \|\varphi^{\frac{3}{2}} e^{s\varphi} u\|^2_{L^2(\Omega)} + s\lambda^2 \|\varphi^{\frac{1}{2}} e^{s\varphi} \nabla_x u\|^2_{L^2(\Omega)} \lesssim \|e^{s\varphi} A u\|^2_{L^2(\Omega)}, \quad u \in \mathscr{C}_c^\infty(\Omega), \ \ s \geq s_0, \ \ \lambda \geq \lambda_0.$$

For a parabolic operator, say $P = \partial_t + \Delta_x$ on $Q = (0, T) \times \Omega$, Carleman estimates can be derived [16] in the following form

$$s^3\lambda^4 \|(a\varphi)^{\frac{3}{2}} e^{sa\eta} u\|^2_{L^2(Q)} + s\lambda^2 \|(a\varphi)^{\frac{1}{2}} e^{sa\eta} \nabla_x u\|^2_{L^2(Q)} \lesssim \|e^{sa\eta} P u\|^2_{L^2(Q)},$$
$$u \in \mathscr{C}^\infty(Q), \ \operatorname{supp}(u(t,.)) \Subset \Omega, \ s \geq s_0, \ \lambda \geq \lambda_0,$$

for $a(t) = (t(T-t))^{-1}$, $\varphi(x) = e^{\lambda\beta(x)}$, with β such that $|\beta'| \neq 0$ and $\eta(x) = e^{\lambda\beta(x)} - e^{\lambda\bar{\beta}} < 0$. In this later case the weight function $a(t)\eta(x)$ is singular at time $t = 0$ and $t = T$. For a review of Carleman estimates for elliptic and parabolic operators we refer to [13, 24].

The estimates we have presented are said to be local, as they apply to compactly supported functions in Ω. So-called global Carleman estimates can be derived (see e.g. [16]). They concern functions defined on the whole Ω with prescribed boundary conditions, e.g. homogeneous Dirichlet, Neumann. They are also characterized by the presence of an observation term on $\omega \subset \Omega$ on the r.h.s. of the estimate, e.g., for the elliptic operator $A = -\Delta$,

$$s^3\lambda^4 \|\varphi^{\frac{3}{2}} e^{s\varphi} u\|^2_{L^2(\Omega)} + s\lambda^2 \|\varphi^{\frac{1}{2}} e^{s\varphi} \nabla_x u\|^2_{L^2(\Omega)} \lesssim \|e^{s\varphi} A u\|^2_{L^2(\Omega)} + s^3\lambda^4 \|\varphi^{\frac{3}{2}} e^{s\varphi} u\|^2_{L^2(\omega)}, \quad s \geq s_0, \ \ \lambda \geq \lambda_0,$$

for $u \in \mathscr{C}^\infty(\overline{\Omega})$, and $u_{|\partial\Omega} = 0$.

Note also that Carleman estimates can be patched together (see e.g. [17, 24]). If local estimates are obtained at the boundary $\partial\Omega$, then one can deduce global estimates from the local ones.

Carleman estimates have many applications. In 1939, T. Carleman introduced these estimates to prove a uniqueness result for some elliptic partial differential equations (PDE) with smooth coefficients in dimension two [10]. This result was later generalized (see e.g. [17, Chapter 8], [18, Chapter 28], [34]). In more recent years, the field of applications of Carleman estimates has gone beyond the original domain. They are also used in the study of inverse problems (see e.g. [9, 20, 19, 22]) and control theory for PDEs. Through unique continuation properties, they are used for the exact controllability of hyperbolic equations [3]. They also yield the null controllability of linear parabolic equations [29] and the null controllability of classes of semi-linear parabolic equations [16, 2, 14].

Difficulties arise for the derivation of Carleman estimates in the case of non-smooth coefficients in the principal part of the operator, i.e., for a regularity lower that Lipschitz. In fact, it is known

[1] $a \lesssim b$ stands for $a \leq Cb$ for some constant $C > 0$.

that unique continuation does not hold in general for a $\mathscr{C}^{0,\alpha}$ Hölder regularity of the coefficient with $0 < \alpha < 1$ [32, 31], which ruins any hope to prove a Carleman estimate.

In the present article, we consider coefficients that are *discontinuous* across a smooth interface, yet regular on each side. This question was first addressed in [12] for a parabolic operator $P = \partial_t - \nabla_x(c(x)\nabla_x)$, with a monotonicity assumption: the observation takes place in the region where the diffusion coefficient c is the 'lowest'. In the one-dimensional case, the monotonicity assumption was relaxed for general piecewise $\mathscr{C}^1$ coefficients [6, 7], and for coefficients with bounded variations [23]. Simultaneously to these results, a controllability result for *linear* parabolic equations with $c \in L^\infty$ was proven in [1] in the one-dimensional case *without* Carleman estimate. This controllability result does not cover more general semi-linear equations. An earlier result was that of [15] where the controllability of a *linear* parabolic equations was proven in one dimension with $c \in BV$ through D. Russel's method.

The case of an arbitrary dimension without any monotonicity condition in the elliptic case was solved in [5, 27] and in the parabolic case in [28]. In [25, 26] the case of a discontinuous anisotropic matrix coefficients is treated and a sharp condition on the weight function is provided for the Carleman estimate to hold.

The methods used in [5, 27, 28, 25, 26] focus on a neighborhood of a point at the interface where the interface can be given by $\{x_n = 0\}$ for an appropriate choice of coordinates $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. Then, through microlocal techniques (Calderón projector or first-order factorization), a local Carleman estimate is proven. However, these methods require strong regularity for the coefficients and for the interface. Moreover, they fall short if the interface crosses the boundary. This configuration is typical in bounded stratified media such as those we consider below.

In stratified media, a controllability result for a *linear* parabolic equation in arbitrary dimension was obtained in [8]. The approach was based on the 1D Carleman estimates of [6, 23] in the *parabolic* case and a spectral inequality as in [29, 30, 21] for the transverse elliptic operator, whose coefficients are smooth. The precise definition of such stratified media is given below.

The controllability result obtained in [8] left the question of deriving a Carleman estimate open for stratified media in dimension greater than two in both the elliptic and the parabolic cases. This result is achieved here. One of the consequences of this result in the parabolic case is the null-controllability of classes of semi-linear parabolic equations. We refer to [12] for these developments.

Remark 1.1. The following observation also provides hints that Carleman estimates can be derived for stratified media [33]. As we shall assume below interfaces cross the boundary transversely. Pick a point at the intersection of an interface and the boundary and choose local coordinates such that the interface is orthogonal to the boundary. Assume that the coefficients associated with the transverse part of the operator are flat at the boundary. Then, by reflection at the boundary, the system under consideration can be turned into a problem with a smooth interface away from any boundary which permits to use the results of [27, 28, 25, 26]. This situation is however not general.

We finish this introductory presentation by pointing out the difficulty that arises when deriving a Carleman estimate for the operator $A = -\nabla_x(c(x)\nabla_x)$ or $P = \partial_t - \nabla_x(c(t,x)\nabla_x)$ in dimension greater than two, in the presence of an interface S. In fact, the standard Carleman derivation method leads to interface terms involving

1. trace of the function $u_{|S}$. Zero- and first-order operators in the tangent direction act on $u_{|S}$.
2. traces of its normal derivative $\partial_{x_n} u_{|S_\pm}$, on both sides of S.

This interface contribution can be interpreted as a quadratic form (see [6]). In [27, 28] the authors show that this quadratic form is only non-negative for low (tangential) frequencies. Here we shall recover this behavior where the tangential Fourier transform is replaced by Fourier series, built on a basis of eigenfunctions of the transverse part of the elliptic operator. For high (tangential) frequencies, the tangential derivative term (i.e., the action of a the first-order operator on $u_{|S}$) yields a negative contribution, unless a monotonicity assumption on the coefficient c is made as in [12]. In [5, 27, 28] the authors have used microlocal methods in the high frequency regime to solve this difficulty, and more recently in [26, 25]. Here, because of the intersection of the interface with the boundary, and because of the little regularity required for the diffusion coefficients, such methods cannot be used directly. However, the separated-variable assumption made on the diffusion coefficients allows us to use Fourier series and similar ideas can be developed: low frequencies and high frequencies are treated differently. In the parabolic case the separation we make between the two frequency regimes is time dependent. Here, the separated-variable assumption yields explicit computations, which reveals the behavior of the solution in each frequency regime.

In the present article, a particular class of anisotropic coefficients is treated. The question of deriving Carleman estimates for more general coefficients in the neighborhood of the intersection of an interface, where the coefficients jump, with the boundary is left open.

1.1 Setting and notation

We let Ω be an open subset in $\mathbb{R}^n$, with $\Omega = \Omega' \times (-H, H)$, where Ω' is a nonempty bounded open subset of $\mathbb{R}^{n-1}$ with $\mathscr{C}^1$ boundary[2].

We shall use the notation $x = (x', x_n) \in \Omega' \times (-H, H)$. We set $S = \Omega' \times \{0\}$, that will be understood as an interface where coefficients and functions may jump. For a function $f = f(x)$ we define by $[f]_S$ its jump at S, i.e.,

$$[f]_S(x') = f(x)_{|x_n=0^+} - f(x)_{|x_n=0^-}.$$

For a function u defined on both sides of S, we set

$$u_{|S_\pm} = (u_{|\Omega_\pm})_{|S},$$

with $\Omega_+ = \Omega' \times (0, H)$ and $\Omega_- = \Omega' \times (-H, 0)$.

Let $B(t, x)$, $t \in (0, T)$ and $x \in \Omega$, be with values in $M_n(\mathbb{R})$, the space of square matrices with real coefficients of order n. We make the following assumption.

[2]Note that the derivation of a Carleman estimate in the case of singular domains can be achieved (see [4]). Addressing the more general case of Lipschitz boundary is an open question to our knowledge.

Assumption 1.2. *The matrix diffusion coefficient $B(t, x', x_n)$ has the following block diagonal form*

$$B(t, x', x_n) = \begin{pmatrix} c_1(t, x_n)C_1(x') & 0 \\ 0 & c_2(t, x_n) \end{pmatrix}$$

where the functions c_i, $i = 1, 2$, are[3] *in $\mathscr{C}^1((0, T)\times\overline{\Omega_\pm})$ with a possible jump at $x_n = 0$. We assume $C_1 \in W^{1,\infty}(\Omega', M_{n-1}(\mathbb{R}))$ and that $C_1(x')$ is hermitian. We further assume uniform ellipticity*

$$\begin{aligned} &0 < c_{\min} \leq c_i(t, x_n) \leq c_{\max} < \infty, \quad (t, x_n) \in (0, T) \times (-H, H) \text{ and } i = 1, 2, \\ &0 < c_{\min}\,\mathrm{Id}_{n-1} \leq C_1(x') \leq c_{\max}\,\mathrm{Id}_{n-1}, \quad x' \in \Omega'. \end{aligned}$$

To lighten notation we shall often write $c_{i-} := c_{i|x_n=0^-}$ and $c_{i+} := c_{i|x_n=0^+}$ for $i = 1, 2$.

Remark 1.3. Here, the matrix coefficient B is chosen time dependent in preparation for the Carleman estimate in the parabolic case. We shall also prove such an estimate in the elliptic case: see Theorem 1.4 below and its proof in Section 3. For this theorem we shall *of course* use B independent of time.

For the proof of Theorem 1.4 (elliptic case) we shall further assume $c_1 = c_2$. In fact, this simplification allows us to provide a fairly simple proof of the Carleman estimates that shows the different treatment of two frequency regimes. These frequency regimes are connected to the microlocal regions used in [27] and [25, 26]. Note however that the case $c_1 \neq c_2$ can also be treated in the elliptic case. The proof is then closer to that of the parabolic case of Theorem 1.5 in Section 4. We have omitted this proof for the sake of the clarity of the exposition.

Let $T > 0$. For each $t \in [0, T]$, we consider the symmetric bilinear H_0^1-coercive form

$$a_t(u, v) = \int_\Omega (B(t, \cdot)\nabla_x u) \cdot \nabla_x v dx,$$

with domain $D(a_t) = H_0^1(\Omega)$. It defines a selfadjoint operator $A_t = -\nabla_x \cdot (B(t, \cdot)\nabla_x)$ in $L^2(\Omega)$ with compact resolvent and with domain $D(A_t) = \{u \in H_0^1(\Omega); \nabla_x \cdot (B(t, \cdot)\nabla_x u) \in L^2(\Omega)\}$ (see e.g. [11], p. 1211). In the elliptic case, we shall denote by $\|\cdot\|_{L^2(\Omega)}$ the L^2 norm over Ω and by $|\cdot|_{L^2(S)}$ the L^2 norm over the interface S of codimension 1.

We set $Q_T = (0, T)\times\Omega$, $S_T = (0, T)\times S$. We shall also consider the following parabolic operator $P = \partial_t + A_t$ on Q_T. In the parabolic case, we shall denote by $\|\cdot\|_{L^2(Q_T)}$ the L^2 norm over Q_T and by $|\cdot|_{L^2(S_T)}$ the L^2 norm over the interface S_T of codimension 1.

In this article, when the constant C is used, it refers to a constant that is independent all the parameters. Its value may however change from one line to another. We shall use the notation $a \lesssim b$ if we have $a \leq Cb$ for such a constant. If we want to keep track of the value of a constant we shall use another letter.

[3]Concerning the regularity of the coefficients c_i, an inspection of the proof of the Carleman estimate in the parabolic case shows that the time derivative of the trace of c_2 at $x_n = 0$ needs to make sense (see above (A.9) in Appendix A.5). An alternative regularity is then $W^{1,\infty}(0, T; W^{1,\infty}(\Omega))$.

1.2 Statements of the main results

We consider ω, a nonempty open subset of Ω. For a function β in $\mathscr{C}^0(\overline{\Omega})$ we set

$$\varphi(x) = e^{\lambda\beta(x)}, \quad \lambda > 0,$$

to be used as weight function. We consider first a matrix coefficient independent of the parameter t. A proper choice of the function β, with respect to the operator A, ω and Ω (see Assumption 2.1 and Assumption 3.2), yields the following Carleman estimate for the elliptic operator A.

Theorem 1.4 (Elliptic case)**.** *There exist $C > 0$, λ_0 and $s_0 > 0$ such that*

$$s\lambda^2 \|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla u\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\Omega)} + s\lambda\Big(|e^{s\varphi}\varphi^{\frac{1}{2}}\nabla_\tau u_{|S}|^2_{L^2(S)} + |e^{s\varphi}\varphi^{\frac{1}{2}}\partial_n u_{|S}|^2_{L^2(S)}\Big)$$
$$+ s^3\lambda^3|e^{s\varphi}\varphi^{\frac{3}{2}}u_{|S}|^2_{L^2(S)} \leq C\Big(\|e^{s\varphi}Au\|^2_{L^2(\Omega)} + s^3\lambda^4\|e^{s\varphi}\varphi^{\frac{3}{2}}u\|^2_{L^2(\omega)}\Big),$$

for all $u \in D(A)$, $\lambda \geq \lambda_0$, and $s \geq s_0$.

Here, ∇_τ is the tangential gradient on the interface S. Note that belonging to the domain $D(A)$ implies some constraints on the function u at the interface S, namely $u \in H^1$ and $B\nabla_x u \in H(\mathrm{div}, \Omega)$. We shall first prove the result for piecewise smooth functions satisfying

$$u_{|S_-} = u_{|S_+}, \quad (c\partial_{x_n}u)_{|S_-} = (c\partial_{x_n}u)_{|S_+},$$

and then use their density in $D(A)$.

With a function $\tilde{\beta} > 0$ that satisfies Assumption 2.1 below, we introduce $\beta = \tilde{\beta} + m\|\tilde{\beta}\|_\infty$ where $m > 1$. For $\lambda > 0$ we define the following weight functions

$$\varphi(x) = e^{\lambda\beta(x)}, \qquad \eta(x) = e^{\lambda\beta(x)} - e^{\lambda\overline{\beta}}, \qquad a(t) = \big(t(T-t)\big)^{-1},$$

with $\overline{\beta} = 2m\|\tilde{\beta}\|_\infty$ (see [16, 12]). For $\tilde{\beta}$ satisfying some additional requirements (Assumption 4.2), that will be provided in Section 4, we prove the following Carleman estimate for the parabolic operator P.

Theorem 1.5 (Parabolic case)**.** *There exist $C > 0$, λ_0 and $s_0 > 0$ such that*

$$s^{-1}\Big(\|e^{sa\eta}(a\varphi)^{-\frac{1}{2}}\partial_t u\|^2_{L^2(Q_T)} + \|e^{sa\eta}(a\varphi)^{-\frac{1}{2}}A_t u\|^2_{L^2(Q_T)}\Big) + s\lambda^2\|e^{sa\eta}(a\varphi)^{\frac{1}{2}}\nabla u\|^2_{L^2(Q_T)}$$
$$+ s^3\lambda^4\|e^{sa\eta}(a\varphi)^{\frac{3}{2}}u\|^2_{L^2(Q_T)} + s\lambda\Big(|e^{sa\eta}(a\varphi)^{\frac{1}{2}}\nabla_\tau u_{|S_T}|^2_{L^2(S_T)} + |e^{sa\eta}(a\varphi)^{\frac{1}{2}}\partial_n u_{|S_T}|^2_{L^2(S_T)}\Big)$$
$$+ s^3\lambda^3|e^{sa\eta}(a\varphi)^{\frac{3}{2}}u_{|S_T}|^2_{L^2(S_T)} \leq C\Big(\|e^{sa\eta}Pu\|^2_{L^2(Q_T)} + s^3\lambda^4\|e^{sa\eta}(a\varphi)^{\frac{3}{2}}u\|^2_{L^2((0,T)\times\omega)}\Big),$$

for all $u \in C^2((0,T)\times\Omega_\pm)$ such that $u_{|S_T^-} = u_{|S_T^+}$, $(c_2\partial_{x_n}u)_{|S_T^-} = (c_2\partial_{x_n}u)_{|S_T^+}$, $\lambda \geq \lambda_0$, and $s \geq s_0(T + T^2)$.

By a density argument, we can extend this estimate to functions in $\int^{\oplus}_{[0,T]} D(A_t)\, dt \cap H^1(0,T; L^2(\Omega))$. Larger function spaces, with rougher behaviors can also be handled such as explosion at times $t = 0$ and $t = T$ if they are compensated by the rapidly vanishing weight function $e^{sa\eta}$. The r.h.s. of the estimate can be used to define a norm. The larger the parameters s and λ, the bigger the associated spaces will be. Such choices can be driven by applications.

1.3 Outline

In Section 2, we provide some spectral properties of operator A, which yields a Hilbert direct decomposition of $L^2(\Omega) = \oplus_{k\in\mathbb{N}^*} H_k$ that reduces A. We also provide the precise assumptions made on the weight function. In Section 3, we prove the Carleman estimate for the elliptic case. In Section 4 we prove the Carleman estimate for a parabolic case. Some intermediate and technical results are collected in the appendices.

2 Spectral properties and weight function

Similarly to $A_t = -\nabla_x \cdot (B(t,x)\nabla_x)$, one can define the time independent selfadjoint transverse operator on $L^2(\Omega')$

$$A' = -\nabla_{x'} \cdot (C_1\nabla_{x'}), \quad D(A') = \{u \in H_0^1(\Omega'); \nabla_{x'} \cdot (C_1\nabla_{x'}u) \in L^2(\Omega')\}.$$

We consider an orthonormal basis of $L^2(\Omega')$, composed of eigenfunctions $(\phi_k)_{k\geq1}$, associated with the eigenvalues, with finite multiplicities, $0 < \mu_1^2 \leq \mu_2^2 \leq \cdots \leq \mu_k^2 \leq \mu_{k+1}^2 \leq \cdots$, with $\mu_k \to \infty$.

With this basis $(\phi_k)_{k\geq1}$, we build an unitary transform $\mathcal{F} : L^2(\Omega) \mapsto \overset{\infty}{\underset{k=1}{\oplus}} L^2(-H,H)$ defined by

$$(\mathcal{F}u)(k,x_n) := \int_{\Omega'} \phi_k(x')u(x',x_n)\,dx', \tag{2.1}$$

with the following properties (recall that here $\nabla_{x'} = \nabla_\tau$)

$$\begin{aligned} v(x',x_n) &= \sum_{k=1}^{\infty} (v(.,x_n),\phi_k)_{L^2(\Omega')}\phi_k(x') = \sum_{k=1}^{\infty} \hat{v}(k,x_n)\phi_k(x'),\\ \nabla_{x'}v(x',x_n) &= \sum_{k=1}^{\infty} \hat{v}(k,x_n)\nabla_{x'}\phi_k(x'). \end{aligned}$$

We shall often write $\hat{v}_k = \hat{v}(k,.)$.

As the family $(C_1^{1/2}\nabla\phi_k)_k$ is orthogonal in $L^2(\Omega')$ (C_1 is a positive definite matrix) we have

$$\|C_1^{1/2}\nabla_{x'}v(.,x_n)\|_{L^2(\Omega')}^2 = \sum_{k=1}^{\infty} |\hat{v}(k,x_n)|^2\, \|C_1^{1/2}\nabla_{x'}\phi_k\|_{L^2(\Omega')}^2 = \sum_{k=1}^{\infty} |\hat{v}(k,x_n)|^2\,\mu_k^2,$$

which gives

$$(c_{\max})^{-1}\sum_{k=1}^{\infty}\mu_k^2\left|\hat{v}(k,x_n)\right|^2 \leq \|\nabla_\tau v(.,x_n)\|_{L^2(\Omega')}^2 \leq (c_{\min})^{-1}\sum_{k=1}^{\infty}\mu_k^2\left|\hat{v}(k,x_n)\right|^2. \tag{2.2}$$

We choose a weight function β that satisfies the following properties.

Assumption 2.1. *The function $\beta \in \mathscr{C}^0(\Omega)$, and $\beta_{|\Omega_\pm} \in \mathscr{C}^2(\overline{\Omega_\pm})$ and*

$$\begin{aligned} &\beta \geq C > 0, \quad |\nabla_x\beta| \geq C > 0 \textit{ in } \Omega\setminus\omega,\\ &\beta = \mathrm{Cst} \textit{ on } \Omega'\times\{-H\} \quad \textit{and} \quad \beta = \mathrm{Cst} \textit{ on } \Omega'\times\{H\}.\\ &\nabla_{x'}\beta = 0 \textit{ on } \partial\Omega'\times(-H,H),\\ &\partial_{x_n}\beta > 0 \textit{ on } \Omega'\times\{-H\}, \quad \textit{and} \quad \partial_{x_n}\beta < 0 \textit{ on } \Omega'\times\{H\}. \end{aligned}$$

There exists a neighborhood V of S in Ω of the form $V = \Omega'\times(-\delta,\delta)$ in which β solely depends on x_n and is a piecewise affine *function of x_n.*

In particular $\beta_{|S}$ is constant. As the open set ω can be shrunk if necessary, we further assume that $\omega \cap (\Omega' \times (-\delta, \delta)) = \emptyset$.

Such a weight function β can be obtained by first designing a function that satisfies the proper properties at the boundaries and at the interface and then construct β by means of Morse functions following the method introduced in [16].

Here, in addition we assume that $\partial_{x_n}\beta = \beta' > 0$ on S_+ and S_-, which means that the observation region ω is chosen in $\Omega' \times (0, H)$, i.e., where $x_n \geq 0$. There is no loss in generality as we can change x_n into $-x_n$ to treat the case of an observation $\omega \subset \Omega' \times (-H, 0)$.

Note that Assumption 2.1 will be completed below by Assumption 3.2 in the elliptic case and Assumption 4.2 in the parabolic case respectively.

3 The elliptic case: proof of Theorem 1.4

As mentioned in the introductory section, we have consider only the case $c_1 = c_2 = c$ in this proof. The case $c_1 \neq c_2$ can be treated following the lines of the proof of Theorem 1.5 in Section 4.

Local Carleman estimates can be stitched together to form a global estimate of the form presented in Theorem 1.4 (see e.g. [24, 28]). Such local estimates are classical away from the interface (see [29, 16, 24]). To prove the elliptic Carleman estimate of Theorem 1.4 it thus remains to prove such a local estimate at the interface S, for functions $u \in D(A)$ with support near the interface. We shall thus assume that $\operatorname{supp}(u) \subset \Omega' \times (-\delta, \delta)$, where the weight function β depends only on x_n and is piecewise affine.

Piecewise smooth functions that satisfy the transmission conditions

$$u_{|S_-} = u_{|S_+}, \quad (c\partial_{x_n} u)_{|S_-} = (c\partial_{x_n} u)_{|S_+}, \tag{3.1}$$

are dense in $D(A)$. We may thus restrict our analysis to such functions. Because of these transmission conditions we shall write $u_{|S}$ and $(c\partial_{x_n} u)_{|S}$ in place of $u_{|S_\pm}$ and $(c\partial_{x_n} u)_{|S_\pm}$ respectively.

Applying the unitary transform of Section 2, the equation $Au = f$ can be written

$$(-\partial_{x_n} c \partial_{x_n} + c\mu_k^2)\hat{u}_k(x_n) = \hat{f}_k(x_n), \quad x_n \in (-\delta, 0) \cup (0, \delta),$$

with $\operatorname{supp} \hat{u}_k \subset (-\delta, \delta)$.

Our starting point is the following proposition.

Proposition 3.1. *Let the weight function β satisfy Assumption 2.1. There exist $C, C', C'' > 0$, $\lambda_0 > 0$, $s_0 > 0$ such that*

$$\begin{aligned} &C\Big(s\lambda^2 \|\varphi^{\frac{1}{2}}\partial_{x_n}\hat{v}_k\|^2_{L^2(-\delta,\delta)} + s\lambda^2\|\varphi^{\frac{1}{2}}\mu_k\hat{v}_k\|^2_{L^2(-\delta,\delta)} + s^3\lambda^4\|\varphi^{\frac{3}{2}}\hat{v}_k\|^2_{L^2(-\delta,\delta)}\Big) \\ &\quad + s\lambda\varphi_{|S}\left([c^2\beta'|\partial_{x_n}\hat{v}_k|^2]_S + |s\lambda\varphi\hat{v}_{k|S}|^2[c^2\beta'^3]_S - |\mu_k\hat{v}_{k|S}|^2[c^2\beta']_S\right) \leq C'\|e^{s\varphi}\hat{f}_k\|^2_{L^2(-\delta,\delta)} + Z, \end{aligned} \tag{3.2}$$

for all $k \in \mathbb{N}^*$, $\hat{v}_k = e^{s\varphi}\hat{u}_k$, $\lambda \geq \lambda_0$ and $s \geq s_0$, with $Z = -C'' s\lambda^2\varphi_{|S} \operatorname{Re}[c^2\beta'^2\partial_{x_n}\hat{v}_k]_S \overline{\hat{v}_{k|S}}$.

We emphasize that the constants are uniform with respect to the transverse-mode index k. Such a result can be obtained by adapting the derivations in [12] for instance. We provide a short proof in Appendix A.1. In particular we have

$$|Z| \leq C s\lambda^2 \varphi_{|S} \Big(|\partial_{x_n} \hat{v}_{k|S_-}| + |\partial_{x_n} \hat{v}_{k|S_+}|\Big)\, |\hat{v}_{k|S}|. \tag{3.3}$$

Moreover, in addition to Assumption 2.1, we shall consider the following particular form of β

Assumption 3.2. *For $K = \frac{c_-}{c_+}$ and some $r \geq 0$, we have*

$$L = \frac{\beta'_{|S_+}}{\beta'_{|S_-}} = \begin{cases} 2 & \text{if } K = 1, \\ K & \text{if } K > 1, \\ (r+1) - rK & \text{if } K < 1. \end{cases} \tag{3.4}$$

Remark 3.3.

1. With this assumption we note that we have $L > 1$ and $L \to 1$ as $K \to 1$, $K \neq 1$. Here we choose $L = 2$ if $K = 1$, to preserve interface terms in the Carleman estimates even for this case that corresponds to coefficients with no jump.

2. The value $r = 3$ is admissible in (3.4) (see Lemma 3.6 and its proof). In the spirit of what is done in [23] one may wish to control the jump of the slope of the weight function by choosing other values for r.

3. To construct the weight function β we first choose its slopes on both sides of the interface satisfying Assumption 3.2. Here the slopes are positive as we wish to observe the solution in $\{x_n > 0\}$. We may then extend the function β on both sides of the interface. The additional requirements of Assumption 2.1 only concern the behavior of the β away from the interface. The two assumptions are compatible.

We now set $\mathcal{B}(v) = s\lambda\varphi_{|S}\Big([c^2\beta'|\partial_{x_n} v|^2]_S + |s\lambda\varphi v_{|S}|^2 [c^2\beta'^3]_S\Big)$.

Lemma 3.4. *We have*

$$\mathcal{B}(\hat{v}_k) = s\lambda\varphi_{|S}\, e^{2s\varphi_{|S}}\Big(B_1|\gamma(\hat{u}_k)|^2 + B_2|s\lambda\varphi\hat{u}_{k|S}|^2\Big), \quad \gamma(\hat{u}_k) = c\partial_{x_n}\hat{u}_{k|S} + c_+\beta'_{|S_-}\frac{L^2 - K}{L-1}(s\lambda\varphi\hat{u}_k)_{|S},$$

with $B_1 = \beta'_{|S_-}(L-1) > 0$, and

$$B_2 = c_+^2(\beta'_{|S_-})^3\Big(2(L^3 - K^2) - \frac{(L^2-K)^2}{L-1}\Big). \tag{3.5}$$

For a proof see Appendix A.2. Note that $L > 1$ by Assumption 3.2.

We shall consider two cases: $K > 1$ and $0 < K \leq 1$.

Case $K > 1$. Then $L = K$ and

$$-[c^2\beta']_S = -c_+^2\beta'_{|S_-}(L - K^2) > 0, \quad B_1 > 0, \quad B_2 = c_+^2(\beta'_{|S_-})^3 K^2(K-1) > 0.$$

The trace terms in (3.2) thus yield a positive contribution. We have

$$\begin{aligned}\mathcal{B}(\hat{v}_k) - s\lambda\varphi_{|S}|\mu_k\hat{v}_{k|S}|^2[c^2\beta']_S &\gtrsim (s\lambda\varphi_{|S})^3 e^{2s\varphi}|\hat{u}_{k|S}|^2 + s\lambda\varphi_{|S}\, e^{2s\varphi}\Big(|\gamma(\hat{u}_k)|^2 + |\mu_k\hat{u}_{k|S}|^2\Big) \\ &\gtrsim (s\lambda\varphi_{|S})^3 e^{2s\varphi}|\hat{u}_{k|S}|^2 + s\lambda\varphi_{|S}\, e^{2s\varphi}\Big(|\partial_{x_n}\hat{u}_{k|S}|^2 + |\mu_k\hat{u}_{k|S}|^2\Big).\end{aligned}$$

In particular for s sufficiently large the remainder term Z estimated in (3.3) can be 'absorbed'. We thus obtain

$$s\lambda^2 \|\varphi^{\frac{1}{2}} \partial_{x_n} \hat{v}_k\|^2_{L^2(-\delta,\delta)} + s\lambda^2 \|\varphi^{\frac{1}{2}} \mu_k \hat{v}_k\|^2_{L^2(-\delta,\delta)} + s^3\lambda^4 \|\varphi^{\frac{3}{2}} \hat{v}_k\|^2_{L^2(-\delta,\delta)} \\ + (s\lambda\varphi_{|S})^3 e^{2s\varphi} |\hat{u}_{k|S}|^2 + s\lambda\varphi_{|S} e^{2s\varphi} \Big(|\partial_{x_n} \hat{u}_{k|S}|^2 + |\mu_k \hat{u}_{k|S}|^2 \Big) \lesssim \|e^{s\varphi} \hat{f}_k\|^2_{L^2(-\delta,\delta)}, \quad (3.6)$$

for all $k \in \mathbb{N}^*$. Summing over k, using (2.2) we obtain the sought local Carleman estimate in the case $K > 1$

$$s\lambda^2 \|\varphi^{\frac{1}{2}} \nabla v\|^2_{L^2(\Omega'\times(-\delta,\delta))} + s^3\lambda^4 \|\varphi^{\frac{3}{2}} v\|^2_{L^2(\Omega'\times(-\delta,\delta))} \\ + (s\lambda\varphi_{|S})^3 e^{2s\varphi_{|S}} |u_{|S}|^2_{L^2(S)} + s\lambda\varphi_{|S} e^{2s\varphi_{|S}} |\nabla u_{|S}|^2_{L^2(S)} \lesssim \|e^{s\varphi} f\|^2_{L^2(\Omega'\times(-\delta,\delta))}. \quad (3.7)$$

The Carleman estimate of Theorem 1.4 can then be deduced classically. This case, $K > 1$ is the case originally covered by [12].

Case $0 < K \leq 1$. Then, either $L = (r+1) - rK > 1$ or $L = 2$, which gives $B_1 > 0$. Lemma 3.6 below implies that $B_2 > 0$. Hence, for s sufficiently large the remainder term Z estimated in (3.3) can be 'absorbed'. We now aim to estimate the tangential term in (3.2).

Proposition 3.5. *There exists $C > 0$, and $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ we have*

$$\Big| s\lambda\varphi_{|S} [c^2\beta']_S \, |\mu_k \hat{v}_{k|S}|^2 \Big| \leq \frac{1}{1+\varepsilon} s\lambda\varphi_{|S} B_2 \left| s\lambda(\varphi\hat{v}_k)_{|S} \right|^2 \\ + C\Big(\|e^{s\varphi} \hat{f}_k\|^2_{L^2(-\delta,\delta)} + s^2\lambda^2 \|\varphi\hat{v}_k\|^2_{L^2(-\delta,\delta)} + \|\partial_{x_n} \hat{v}_k\|^2_{L^2(-\delta,\delta)} \Big). \quad (3.8)$$

Proof. Let $0 < \varepsilon < 1$. The value of ε will be determined below. We treat low and high values of μ_k differently.

Low frequencies. Set k_1 as the largest integer such that $(1+\varepsilon)\left|[c^2\beta']_S\right| \mu_k^2 < B_2 (s\lambda\varphi)^2_{|S}$, that is

$$(1+\varepsilon)\mu_k^2 < (\beta'_{|S_-})^2 (s\lambda\varphi)^2_{|S} \frac{1}{|L - K^2|} \Big(2(L^3 - K^2) - \frac{(L^2 - K)^2}{L - 1} \Big). \quad (3.9)$$

We then have

$$(1+\varepsilon) s\lambda\varphi_{|S} \left| [c^2\beta']_S \right| |\mu_k \hat{v}_{k|S}|^2 < s\lambda\varphi_{|S} B_2 \left| s\lambda(\varphi\hat{v}_k)_{|S} \right|^2, \quad k \leq k_1. \quad (3.10)$$

High frequencies. Here we consider frequencies μ_k that satisfy

$$(1-\varepsilon)\mu_k \geq s|\partial_{x_n}\varphi_{|S_-}| = s\lambda\varphi_{|S}\beta'_{|S_-}. \quad (3.11)$$

We denote by k_2 the smallest integer that satisfies (3.11).

We write

$$(\partial^2_{x_n} - \mu_k^2)\hat{u}_k = -\frac{\hat{f}_k}{c} - \frac{\partial_{x_n} c}{c} \partial_{x_n} \hat{u}_k = -\hat{g}_k.$$

As $\hat{u}_k(-\delta) = \hat{u}_k(\delta) = 0$, with the transmission conditions (3.1), the computations[4] of Appendix A.3 yield

$$\mu_k \hat{u}_k|_{x_n=0^+} = \frac{1}{(c_+ + c_-)} \int_0^\delta \frac{\sinh(\mu_k(\delta - x_n))}{\cosh(\mu_k \delta)} (c_+ \hat{g}_k(x_n) + c_- \hat{g}_k(-x_n))\, dx_n. \tag{3.12}$$

We have

$$\frac{\sinh(\mu_k(\delta - x_n))}{\cosh(\mu_k \delta)} = \frac{e^{\mu_k(\delta - x_n)} - e^{-(\mu_k(\delta - x_n))}}{e^{\mu_k \delta} + e^{-(\mu_k \delta)}} \le e^{-\mu_k x_n}. \tag{3.13}$$

We note that

$$\varphi(0) - \varphi(-x_n) = x_n \int_0^1 \varphi'(-x_n + \sigma x_n)\, d\sigma = x_n \lambda \beta'_{|S_-} \int_0^1 \varphi(-x_n + \sigma x_n)\, d\sigma,$$

as the weight function $\beta = \beta(x_n)$ is affine in $(-\delta, 0)$. Since $\beta' > 0$, the function φ increases with x_n and we have $\varphi(0) \le \varphi(-x_n) + x_n \lambda \varphi(0) \beta'_{|S_-}$, if $x_n > 0$. As we have assumed (3.11) here we obtain

$$s\varphi(0) - \mu_k x_n \le s\varphi(-x_n) - \varepsilon \mu_k x_n, \quad x_n > 0. \tag{3.14}$$

We also have

$$s\varphi(0) - \mu_k x_n \le s\varphi(x_n) - \varepsilon \mu_k x_n, \quad x_n > 0. \tag{3.15}$$

From (3.12) we thus obtain

$$\begin{aligned}
\mu_k^{\frac{3}{2}} e^{s\varphi_{|S}} |\hat{u}_{k|S}| &\le \frac{1}{(c_+ + c_-)} \int_0^\delta \Big(e^{s\varphi(-x_n)} |c_- \hat{g}_k(-x_n)| + e^{s\varphi(x_n)} |c_+ \hat{g}_k(x_n)| \Big) \mu_k^{\frac{1}{2}} e^{-\varepsilon \mu_k x_n}\, dx_n \\
&\lesssim \Big(\|e^{s\varphi} \hat{g}_k\|_{L^2(-\delta,0)} + \|e^{s\varphi} \hat{g}_k\|_{L^2(0,\delta)} \Big) \Big(\int_0^\delta \mu_k e^{-2\varepsilon \mu_k x_n}\, dx_n \Big)^{\frac{1}{2}} \\
&\lesssim \varepsilon^{-\frac{1}{2}} \|e^{s\varphi} \hat{g}_k\|_{L^2(-\delta,\delta)} \lesssim \varepsilon^{-\frac{1}{2}} \Big(\|e^{s\varphi} \hat{f}_k\|_{L^2(-\delta,\delta)} + s\lambda \|\varphi \hat{v}_k\|_{L^2(-\delta,\delta)} + \|\partial_{x_n} \hat{v}_k\|_{L^2(-\delta,\delta)} \Big),
\end{aligned}$$

which leads to, for $k \ge k_2$,

$$\begin{aligned}
&s\lambda \varphi_{|S} \left| [c^2 \beta']_S \right| |\mu_k \hat{v}_{k|S}|^2 \\
&\quad \lesssim (1 - \varepsilon) |\beta'_-|^{-1} \left| [c^2 \beta']_S \right| \mu_k^3 |\hat{v}_{k|S}|^2 \\
&\quad \lesssim (1 - \varepsilon) \varepsilon^{-1} |\beta'_-|^{-1} \left| [c^2 \beta']_S \right| \Big(\|e^{s\varphi} \hat{f}_k\|^2_{L^2(-\delta,\delta)} + s^2 \lambda^2 \|\varphi \hat{v}_k\|^2_{L^2(-\delta,\delta)} + \|\partial_{x_n} \hat{v}_k\|^2_{L^2(-\delta,\delta)} \Big).
\end{aligned}$$

We have thus seen that low frequencies in (3.8) are estimated by boundary terms and high frequencies are estimated by the r.h.s. of (3.2) and "absorbable" terms. It remains to prove that we cover the whole spectrum with the two estimates we have obtained. A sufficient condition is then

$$(1 - \varepsilon)^{-2} (s\lambda \varphi_{|S})^2 (\beta'_{|S_-})^2 \le \frac{1}{1 + \varepsilon} (\beta'_{|S_-})^2 (s\lambda \varphi)^2_{|S} \frac{1}{|L - K^2|} \Big(2(L^3 - K^2) - \frac{(L^2 - K)^2}{L - 1} \Big),$$

[4]This is the precise point where $c_1 = c_2$ is used. In the case $c_1 \ne c_2$ the result of Appendix A.3 cannot be used and we have to proceed as in Section 4.

that is

$$P(K,L) := -|L - K^2|(L-1) + \frac{(1-\varepsilon)^2}{1+\varepsilon}\Big(2(L^3 - K^2)(L-1) - (L^2 - K)^2\Big) \geq 0. \tag{3.16}$$

We recall that $L = (r+1) - rK$ if $0 < K < 1$. The following lemma provides a positive answer (see Appendix 3.6 for a proof).

Lemma 3.6. *There exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$,

- $P(K,L) \geq 0$ *if* $K = 1$,
- *there exists* $r \geq 1$ *such that* $P(K,L) \geq 0$ *for* $K \in (0,1)$. *In particular the value* $r = 3$ *is admissible.*

In particular we have $B_2 > 0$.

This concludes the proof of Proposition 3.5. ∎

Arguing as we did for (3.7) in the case $K > 1$, we now obtain

$$\begin{aligned} &s\lambda^2 \|\varphi^{\frac{1}{2}} \nabla v\|^2_{L^2(\Omega' \times (-\delta,\delta))} + s^3\lambda^4 \|\varphi^{\frac{3}{2}} v\|^2_{L^2(\Omega' \times (-\delta,\delta))} + \big(s\lambda\varphi_{|S}\big)^3 e^{2s\varphi_{|S}} |u_{|S}|^2_{L^2(S)} \\ &+ s\lambda\varphi_{|S} e^{2s\varphi_{|S}} |\nabla u_{|S}|^2_{L^2(S)} \lesssim \|e^{s\varphi} f\|^2_{L^2(\Omega' \times (-\delta,\delta))} + s^2\lambda^2 \|\varphi v\|^2_{L^2(\Omega' \times (-\delta,\delta))} + \|\partial_{x_n} v\|^2_{L^2(\Omega' \times (-\delta,\delta))}. \end{aligned} \tag{3.17}$$

The last two terms on the r.h.s. can be "absorbed" by the l.h.s. by choosing s sufficiently large. This concludes the case $0 < K \leq 1$ and the proof of Theorem 1.4.

Remark 3.7. Here, the weight function does not depend on x'. Observe that the local Carleman estimate that we obtain in $\Omega' \times (-\delta,\delta)$ does not require any regularity for the boundary of the open set Ω'. The minimal regularity of the boundary $\partial\Omega$ to achieve a Carleman estimate remains an open question to our knowledge.

4 The parabolic case: proof of Theorem 1.5

Here, the matrix coefficient B is assumed to be time dependent as stated in Assumption 1.2. The coefficients $c_1(t,x_n)$ and $c_2(t,x_n)$ can be different.

We choose a function $\tilde{\beta} > 0$ that satisfies the requirements of Assumption 2.1 and we introduce $\beta = \tilde{\beta} + m\|\tilde{\beta}\|_\infty$ where $m > 1$. Observe that β also satisfies Assumption 2.1.

For $T > 0$ and $\lambda > 0$ we define the following weight functions

$$\varphi(x) = e^{\lambda\beta(x)}, \qquad \eta(x) = e^{\lambda\beta(x)} - e^{\lambda\bar{\beta}}, \quad x \in \Omega, \qquad a(t) = (t(T-t))^{-1}, \quad t \in (0,T), \tag{4.1}$$

with $\bar{\beta} = 2m\|\tilde{\beta}\|_\infty$ (see [12]). Note that $\eta < 0$. As in the previous sections we choose $\beta' > 0$ on S_+ and S_-, which means that the observation region ω is chosen in $\Omega' \times (0,H)$, i.e., where $x_n \geq 0$. It suffices to prove a local Carleman estimate at the interface S, i.e., for functions u with support near the interface, $\mathrm{supp}(u) \subset [0,T] \times \Omega' \times (-\delta,\delta)$, where the weight function β depends only on x_n and is piecewise affine.

We assume moreover that u satisfies the transmission conditions

$$u_{|S_T^-} = u_{|S_T^+}, \quad (c_2\partial_{x_n}u)_{|S_T^-} = (c_2\partial_{x_n}u)_{|S_T^+}. \tag{4.2}$$

Applying the unitary transform of Section 2, the equation $\partial_t u + Au = f$ can be written

$$(\partial_t - \partial_{x_n}c_2\partial_{x_n} + c_1\mu_k^2)\hat{u}_k(t,x_n) = \hat{f}_k(t,x_n), \quad t \in (0,T), \quad x_n \in (-\delta,0)\cup(0,\delta), \quad k \geq 1,$$

with $\operatorname{supp}(\hat{u}_k) \subset [0,T]\times(-\delta,\delta)$. Setting $q_{T,\delta} = (0,T)\times(-\delta,\delta)$, our starting point is the following proposition.

Proposition 4.1. *Let $T > 0$. There exist $C, C' > 0$, $\lambda_0 > 0$, $s_0 > 0$ such that*

$$\begin{aligned} C\Big(s^{-1}\|(a\varphi)^{-\frac{1}{2}}\partial_t\hat{v}_k\|^2_{L^2(q_{T,\delta})} + s\lambda^2\|(a\varphi)^{\frac{1}{2}}\partial_{x_n}\hat{v}_k\|^2_{L^2(q_{T,\delta})} + s\lambda^2\|(a\varphi)^{\frac{1}{2}}\mu_k\hat{v}_k\|^2_{L^2(q_{T,\delta})} \\ + s^3\lambda^4\|(a\varphi)^{\frac{3}{2}}\hat{v}_k\|^2_{L^2(q_{T,\delta})}\Big) + s\lambda\int_0^T a\varphi_{|S}\left([c_2^2\beta'|\partial_{x_n}\hat{v}_k|^2]_S + |s\lambda a(\varphi\hat{v}_k)_{|S}|^2[c_2^2\beta'^3]_S\right)dt \\ \leq C'\|e^{sa\eta}\hat{f}_k\|^2_{L^2(q_{T,\delta})} + s\lambda\int_0^T a\varphi_{|S}\ |\mu_k\hat{v}_{k|S_T}|^2[c_1c_2\beta']_S\ dt + Z, \end{aligned} \tag{4.3}$$

for all $k \in \mathbb{N}^*$, $\hat{v}_k = e^{sa\eta}\hat{u}_k$, $\lambda \geq \lambda_0$ and $s \geq s_0(T+T^2)$, with

$$|Z| \lesssim s^{\frac{1}{2}}\lambda T\int_0^T a\varphi_{|S}(|\partial_{x_n}\hat{v}_k|^2_{|S_T^-} + |\partial_{x_n}\hat{v}_k|^2_{|S_T^+})\,dt + \left(s(T^3+T^4)\lambda + s^{\frac{3}{2}}T^3\lambda^3\right)\int_0^T a^3\varphi_{|S}|\hat{v}_k|^2_{|S_T}\,dt. \tag{4.4}$$

We emphasize that the constants are uniform with respect to the transverse-mode index k. Such a result can be obtained by adapting the derivations in [12] for instance. We provide a short proof in Appendix A.5.

As in Section 3, we set

$$\mathcal{B}_p(\hat{v}_k) = s\lambda a\varphi_{|S}\left([c_2^2\beta'|\partial_{x_n}\hat{v}_k|^2]_S + |s\lambda a\varphi\hat{v}_{k|S_T}|^2[c_2^2\beta'^3]_S\right),$$

$$L = \frac{\beta'_{|S_+}}{\beta'_{|S_-}}, \quad K_i(t) = \frac{c_{i-}(t)}{c_{i+}(t)}, \qquad \underline{K}_i = \inf_{t\in[0,T]} K_i(t), \quad \overline{K}_i = \sup_{t\in[0,T]} K_i(t),\ i = 1,2. \tag{4.5}$$

and

$$\mathsf{B} = \mathsf{B}(L) = \inf_{t\in[0,T]}(c_{2+}^2(t))\,(\beta'_{|S_-})^3\frac{\underline{K}_2^2 + L^3(L-\underline{L})}{L-1}, \quad \text{with } \underline{L} = \max\{\overline{K}_2, 2\}, \tag{4.6}$$

and finally

$$\mathsf{D} = \mathsf{D}(L) = \sup_{t\in[0,T]}(c_{1+}c_{2+})(t)\beta'_{|S_-}(L + \overline{K}_1\,\overline{K}_2) > 0. \tag{4.7}$$

We make the following assumption on the weight function in addition to Assumption 2.1.

Assumption 4.2. *The weight function β is chosen such that $L \geq \underline{L} = \max\{\overline{K}_2, 2\}$ and*

$$\frac{1}{2} \geq \max\left\{2\sqrt{\frac{\mathsf{D}}{\mathsf{B}}}, \frac{4\beta'_{-|S}}{\sigma}\sqrt{\frac{\mathsf{D}}{\mathsf{B}}}\right\}, \qquad \sigma = \left(\inf_{t,x_n}\frac{c_1(t,x_n)}{c_2(t,x_n)}\right)^{\frac{1}{2}}, \tag{4.8}$$

The coefficients c_1, c_2 being fixed, the forms of the coefficients D and B show that this can be achieved by first choosing the value of $\beta'_{|S_-} > 0$ and then picking a sufficiently large value for L.

Remark 4.3. To construct the weight function β we first choose its slopes on both sides of the interface satisfying Assumption 4.2. Here the slopes are positive as we wish to observe the solution in $\{x_n > 0\}$. We may then extend the function β on both sides of the interface. The additional requirements of Assumption 2.1 only concern the behavior of the β away from the interface. The two assumptions are compatible.

Lemma 4.4. *We have*

$$\mathcal{B}_p(\hat{v}_k) = s\lambda a\varphi_{|S}\, e^{2sa\varphi_{|S}} \Big(B_1|\gamma(\hat{u}_k)|^2 + B_2|s\lambda a(\varphi\hat{u}_k)_{|S}|^2\Big),$$

with $\gamma(\hat{u}_k) = (c_2\partial_{x_n}\hat{u}_k)_{|S} + c_{2+}\beta'_{|S_-}\frac{L^2-K_2}{L-1}(s\lambda a\varphi\hat{u}_k)_{|S}$ *and where*

$$B_1 = \beta'_{|S_-}(L-1), \qquad B_2(t) = c_{2+}^2(t)(\beta'_{|S_-})^3\Big(2(L^3 - K_2^2(t)) - \frac{(L^2 - K_2(t))^2}{L-1}\Big).$$

If β satisfies Assumption 4.2 we have $B_1 > 0$ and $B_2(t) \geq \mathsf{B}$, with B defined in (4.6).

Proof. The proof of Lemma 3.4 in Appendix A.2 can be directly adapted and gives the first part of the lemma. As $L \geq 1$ we have $B_1 > 0$. A direct computation yields $B_2(t) = c_{2+}^2(t)(\beta'_{|S_-})^3\frac{P_p(L,K_2(t))}{L-1}$ with

$$\begin{aligned} P_p(L, Y) &= Y^2(1-2L) + 2YL^2 + L^4 - 2L^3 \\ &= L^3(L - \underline{L}) + L^3(\underline{L} - 2) + 2LY(L - Y) + Y^2. \end{aligned}$$

As $\underline{L} \geq 2$, and $L \geq \overline{K}_2 \geq K_2(t) \geq \underline{K}_2 > 0$, we thus obtain $P_p(L, K_2(t)) \geq \underline{K}_2^2 + L^3(L - \underline{L})$. ■

We now prove the following key result, providing an estimate of the tangential derivative of v, i.e., $\mu_k\hat{v}_k$, in the Fourier decomposition.

Proposition 4.5. *For a weight function β that satisfies Assumptions 2.1 and 4.2 there exists $C > 0$ such that for all $k \in \mathbb{N}^*$ we have*

$$\begin{aligned} s\lambda \int_0^T a\varphi_{|S}|[c_1c_2\beta']_S|\,|\mu_k\hat{v}_{k|S}|^2 dt \leq{}& \frac{\mathsf{B}}{4}(s\lambda)^3|(a\varphi_{|S})^{\frac{3}{2}}\hat{v}_k)_{|S}|^2_{L^2((0,T))} + C\Big(\|e^{sa\eta}\hat{f}_k\|^2_{L^2(q_{T,\delta})} \\ &+ s^3\lambda^3\|(a\varphi)^{\frac{3}{2}}\hat{v}_k\|^2_{L^2(q_{T,\delta})} + s\lambda\|(a\varphi)^{\frac{1}{2}}\partial_{x_n}\hat{v}_k\|^2_{L^2(q_{T,\delta})} + s\lambda\|(a\varphi)^{\frac{1}{2}}\mu_k\hat{v}_k\|^2_{L^2(q_{T,\delta})}\Big). \end{aligned} \tag{4.9}$$

for λ and $s/(T + T^2)$ both sufficiently large.

Proof. We fix $k \geq 1$ and we shall keep track of the dependency of the constants on k.

We have

$$\Big|[c_1c_2\beta']_S\Big| \leq (c_{1+}c_{2+})(t)\,\beta'_{|S_-}(L + K_1K_2(t)) \leq \mathsf{D},$$

with D as defined in (4.7). We set

$$\Phi(t; s, \lambda) := \frac{1}{2}s\lambda a(t)\varphi_{|S}\sqrt{\frac{\mathsf{B}}{\mathsf{D}}} \quad \text{and} \quad \mu_{s,\lambda} := \Phi\Big(\frac{T}{2}; s, \lambda\Big) = \min_{t\in(0,T)} \Phi(t; s, \lambda). \tag{4.10}$$

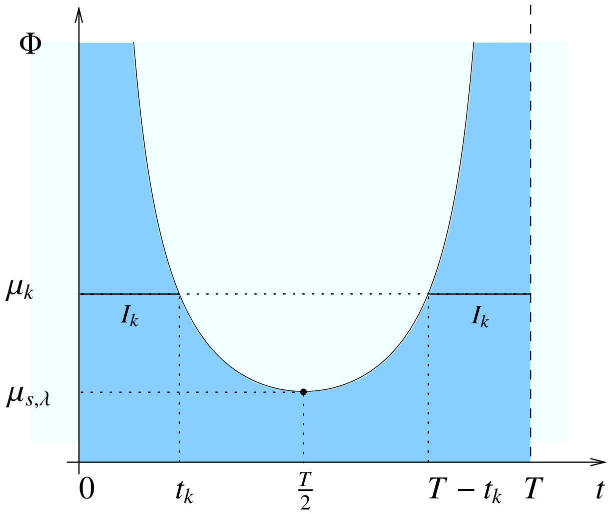

Figure 1: The function Φ. The shaded region is treated in the first step of the proof.

If $\mu_k > \mu_{s,\lambda}$, there exists $t_k := t_k(s, \lambda) \in (0, T/2)$ such that

$$\mu_k = \Phi(t_k; s, \lambda) = \Phi(T - t_k; s, \lambda). \tag{4.11}$$

For $\mu_k \geq \mu_{s,\lambda}$, we set

$$I_k := (0, t_k) \cup (T - t_k, T), \qquad J_k := (0, T) \setminus I_k = (t_k, T - t_k), \qquad \tilde{J}_k := \Big(\frac{t_k}{2}, T - \frac{t_k}{2}\Big).$$

For $\mu_k < \mu_{s,\lambda}$, we set

$$I_k := (0, T).$$

We then introduce

$$I(k; s, \lambda) := s\lambda \mathsf{D} \int_{I_k} a(t)\varphi_{|S}\mu_k^2 |\hat{v}_{k|S}|^2 \, dt, \tag{4.12}$$

$$J(k; s, \lambda) := s\lambda \mathsf{D} \int_{J_k} a(t)\varphi_{|S}\mu_k^2 |\hat{v}_{k|S}|^2 \, dt, \tag{4.13}$$

so that the term on the l.h.s. of (4.9) is less than the sum of the two previous quantities.

The first term, $I(k; s, \lambda)$, involving time t close to 0 or T, will be estimated by a trace term. The second term, $J(k; s, \lambda)$, involving time t away from 0 and T, will be estimated by volume terms.

Step 1: $\mu_k \leq \mu_{s,\lambda}$ or $t \in I_k$ In the (t, μ_k) plane presented in Figure 1 this corresponds to the shaded region. We thus treat low (tangential) frequencies here.

Lemma 4.6. *For all $k \geq 1$ we have*

$$\mathsf{D}|\mu_k \hat{v}_{k|S}|^2 \leq \frac{\mathsf{B}}{4} \left| s\lambda a(t)(\varphi \hat{v}_k)_{|S} \right|^2,$$

with B *as defined in* (4.6)*, if either (1)* $\mu_k \leq \mu_{s,\lambda}$ *or (2)* $\mu_k > \mu_{s,\lambda}$ *and* $t \in I_k$.

Proof. The first point follows from the definition of $\mu_{s,\lambda}$. The second point is a direct consequence of the definition of t_k in (4.11) as the function $t \mapsto a(t)$ decreases on $(0, T/2)$. ■

For all $k \in \mathbb{N}^*$, we thus obtain

$$I(k; s, \lambda) \leq \frac{\mathsf{B}}{4}(s\lambda)^3 |(a\varphi_{|S})^{\frac{3}{2}} \hat{v}_k)_{|S}|^2_{L^2((0,T))}.$$

Step 2: $\mu_k > \mu_{s,\lambda}$ and t in a neighborhood of J_k, preliminary result. In each open set $(0, T) \times (-\delta, 0)$ and $(0, T) \times (0, \delta)$, the function $\hat{u}_k$ satisfies the following equation

$$-\partial^2_{x_n} \hat{u}_k + \frac{c_1}{c_2}\mu_k^2 \hat{u}_k + \frac{1}{c_2}\partial_t \hat{u}_k = \frac{\hat{f}_k}{c_2} + \frac{\partial_{x_n} c_2}{c_2}\partial_{x_n} \hat{u}_k. \tag{4.14}$$

Because of the form of (4.13) we set

$$p(t; s, \lambda) := s\lambda \mathsf{D} a(t)\varphi_{|S}\, e^{2sa(t)\eta_{|S}}. \tag{4.15}$$

We consider a cutoff function $(0, T) \ni t \to \chi_k(t)$, such that

$$\chi_k \equiv 1 \ \text{ on } J_k, \quad 0 \leq \chi_k \leq 1, \quad \operatorname{supp}(\chi_k) \subset \tilde{J}_k \quad \text{and} \quad \|\chi_k'\|_\infty \leq C/t_k,$$

and we introduce

$$w = w(t, k, x_n; s, \lambda) = \frac{1}{2}\chi_k(t) p(t; s, \lambda) |\hat{u}_k(t, x_n)|^2. \tag{4.16}$$

Note that χ_k depends on the index k. Yet, as this dependency will only appear below through the estimate of $\|\chi_k'\|_\infty$ we shall write χ in place of χ_k for concision.

Observe that $w \geq 0$ and that it satisfies the same transmission conditions (4.2) as u. The function w satisfies

$$\partial^2_{x_n} w - \frac{c_1}{c_2}\left((2 - \gamma)\mu_k^2 - \frac{p'}{c_1 p}\right) w = -g, \tag{4.17}$$

with $0 < \gamma < 1$ and

$$g = -\chi p |\partial_{x_n} \hat{u}_k|^2 - \frac{1}{c_2}\partial_t w + \chi \frac{1}{c_2} p \operatorname{Re} \hat{f}_k \overline{\hat{u}_k} - \frac{c_1}{c_2}\mu_k^2 \frac{\gamma}{2}\chi p |\hat{u}_k|^2 + \chi p \frac{\partial_{x_n} c_2}{c_2} \operatorname{Re} \hat{u}_k \partial_{x_n} \overline{\hat{u}_k} + \frac{\chi'}{2c_2} p |\hat{u}_k|^2.$$

Lemma 4.7. *There exist $s_0 > 0$, $\lambda_0 > 0$, depending on L and γ, such that*

$$(2 - \gamma)\mu_k^2 - \frac{p'}{c_1 p} \geq \mu_k^2 \ \ \textit{if} \ \ \frac{t_k}{2} < t < T - \frac{t_k}{2}, \ \ x_n \in (-\delta, \delta), \tag{4.18}$$

for $s > s_0(T + T^2)$ and $\lambda > \lambda_0$.

See Appendix A.6 for a proof.

Step 3: $\mu_k > \mu_{s,\lambda}$ and t in a neighborhood of J_k, conclusion. For $t \in \tilde{J}_k$ we begin by replacing the time-space dependent coefficient $\frac{c_1}{c_2}\Big((2-\gamma)\mu_k^2 - \frac{p'}{c_1 p}\Big)$ by $\sigma^2\mu_k^2$ on the l.h.s. of (4.17) (the constant σ is introduced in (4.8)). This will allow us to argue as in the elliptic case, viz. solving an ordinary differential equation with constant coefficients.

We set

$$q(t, x_n; k, s, \lambda) := -\sigma^2\mu_k^2 + \frac{c_1}{c_2}\Big((2-\gamma)\mu_k^2 - \frac{p'}{c_1 p}\Big).$$

We have

$$\partial_{x_n}^2 w - \sigma^2\mu_k^2 w = -\tilde{g}, \tag{4.19}$$

with

$$\tilde{g} := -qw - \chi p|\partial_{x_n}\hat{u}_k|^2 - \frac{1}{c_2}\partial_t w + \chi\frac{1}{c_2}p \operatorname{Re} \hat{f}_k\overline{\hat{u}_k} - \frac{c_1}{c_2}\chi\frac{\gamma}{2}p\mu_k^2|\hat{u}_k|^2 + \chi p\frac{\partial_{x_n}c_2}{c_2}\operatorname{Re} \hat{u}_k\partial_{x_n}\overline{\hat{u}_k} + \frac{\chi'}{2c_2}p|\hat{u}_k|^2.$$

Observe that Lemma 4.7 gives

$$q(t, x_n; k, s, \lambda) \geq 0, \qquad \frac{t_k}{2} < t < T - \frac{t_k}{2}, \quad x_n \in (-\delta, \delta), \quad s > s_0, \quad \lambda > \lambda_0.$$

From (4.19) and Appendix A.3 we obtain

$$\begin{aligned} J(k; s, \lambda) &\leq 2\int_{\tilde{J}_k} \mu_k^2 w_{|S}\, dt \\ &= \frac{2\mu_k}{\sigma}\int_{\tilde{J}_k}\int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{(c_{2+} + c_{2-})\cosh(\sigma\mu_k\delta)}(c_{2+}\tilde{g}(t, x_n) + c_{2-}\tilde{g}(t, -x_n))\, dx_n dt. \end{aligned} \tag{4.20}$$

Note that the introduction of $\tilde{J}_k$, instead of J_k, is due to the cut-off function χ. Substituting $\tilde{g}$ in (4.20) we obtain seven terms. We shall provide the details for the contribution of $c_{2+}\tilde{g}(t, x_n)$. For the contribution $c_{2-}\tilde{g}(t, -x_n)$ details are given if difference occurs. As in the elliptic case, we shall use that the kernel $e^{-2sa(t)\eta_{|S}}\frac{\sinh(\sigma\mu_k(\delta - x_n))}{\cosh(\sigma\mu_k\delta)}$ be estimated by the weight $e^{2sa\eta}$.

1. We have
$$\int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{\cosh(\sigma\mu_k\delta)}\big(-qw - \chi p|\partial_{x_n}\hat{u}_k|^2\big)\, dx_n \leq 0. \tag{4.21}$$
The negative sign is fortunate as the absolute value of this term cannot be reasonably bounded, i.e., by a term that can be "absorbed" by the l.h.s. of (4.3).

2. (a) **Term $\frac{1}{c_2}\partial_t w$.** Because of the cut-off function χ we have $w_{|t=t_k/2} = w_{|t=T-t_k/2} = 0$ and we get
$$\begin{aligned} &\int_{\tilde{J}_k}\int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{(c_{2+} + c_{2-})\cosh(\sigma\mu_k\delta)}\Big(-\frac{c_{2\pm}}{c_2}\partial_t w(t, \pm x_n)\Big)\, dx_n\, dt \\ &\quad = \int_{\tilde{J}_k}\int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{\cosh(\sigma\mu_k\delta)}\partial_t\Big(\frac{c_{2\pm}}{(c_{2+} + c_{2-})c_2}\Big)w(t, \pm x_n)\, dx_n\, dt, \end{aligned}$$

and, by (3.13), we have

$$\Big|\mu_k \int_{\tilde J_k}\int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{(c_{2+} + c_{2-})\cosh(\sigma\mu_k\delta)}\Big(- \frac{c_{2\pm}}{c_2}\partial_t w(t, \pm x_n)\Big)\, dx_n\, dt\Big|$$
$$\lesssim s\lambda \mathsf{D} \int_{\tilde J_k}\int_0^\delta a(t)\varphi_{|S}\, e^{-\sigma\mu_k x_n} e^{2sa(t)\eta_{|S}} \mu_k |\hat u_k(t, \pm x_n)|^2\, dx_n\, dt.$$

We shall thus obtain an estimate of this term by the r.h.s. of (4.9) if we prove

$$-\sigma\mu_k x_n + 2sa(t)\eta_{|S} \le 2sa(t)\eta(\pm x_n), \quad \forall (t, x_n) \in \tilde J_k \times (0, \delta). \tag{4.22}$$

This is clear for the case + since $\eta_{|S} \le \eta(x_n)$.
The argument is different for the case −. Using that β is a piecewise affine, we have

$$\eta(-x_n) - \eta_{|S} = \varphi(-x_n) - \varphi_{|S} \ge -x_n\lambda(\beta'_-\varphi)_{|S}, \quad x_n \in (0, \delta).$$

Therefore, (4.22) will be satisfied if

$$\sigma\mu_k \ge 2sa(t)\lambda(\beta'_-\varphi)_{|S}, \quad \forall t \in \tilde J_k,$$

which, by the definition of Φ in (4.10), can be written as

$$\sigma\mu_k \ge 4\beta'_{-|S}\sqrt{\frac{\mathsf{D}}{\mathsf{B}}}\Phi(t; s, \lambda), \qquad \forall t \in \tilde J_k.$$

As $\max_{t\in\tilde J_k}\Phi(t; s, \lambda) = \Phi\big(\frac{t_k}{2}; s, \lambda\big)$, it suffices to have

$$\sigma\mu_k = \sigma\Phi(t_k; s, \lambda) \ge 4\beta'_{-|S}\sqrt{\frac{\mathsf{D}}{\mathsf{B}}}\Phi\Big(\frac{t_k}{2}; s, \lambda\Big), \qquad \forall t \in \tilde J_k.$$

This holds if we have

$$\frac{a(t_k)}{a(\frac{t_k}{2})} \ge \frac{4\beta'_{-|S}}{\sigma}\sqrt{\frac{\mathsf{D}}{\mathsf{B}}}. \tag{4.23}$$

Observing that $\frac{a(t_k)}{a(t_k/2)} \ge \frac{1}{2}$, we find that (4.23) is fulfilled by Assumption 4.2.

(b) **Term $\chi(\frac{1}{c_2} p \operatorname{Re} \hat f \overline{\hat u_k} - \frac{c_1}{c_2}\frac{\gamma}{2} p\mu_k^2 |\hat u_k|^2)$.**
We shall prove that the associated term in (4.20) is estimated by $\|e^{sa\eta} f\|^2_{L^2(q_{T,\delta})}$. Applying the Young inequality, we obtain

$$\frac{\mu_k p \operatorname{Re} \hat f_k \overline{\hat u_k}}{c_2} \le \frac{\mathsf{D}}{2\gamma \inf_{t\in[0,T]}(c_1 c_2)} |e^{sa\eta_{|S}}\hat f_k|^2 + \frac{c_1}{c_2}\frac{\gamma s\lambda a\varphi_{|S}\, p\mu_k^2 |\hat u_k|^2}{2}. \tag{4.24}$$

Observe that

$$\mu_k \ge s\lambda a(t)\varphi_{|S}, \;\; t \in \tilde J_k \;\; \Leftrightarrow \;\; \mu_k \ge 2\sqrt{\frac{\mathsf{D}}{\mathsf{B}}}\Phi(t; s, \lambda), t \in \tilde J_k.$$

Arguing as above this will be fulfilled if

$$\frac{a(t_k)}{a(\frac{t_k}{2})} \geq 2\sqrt{\frac{\mathsf{D}}{\mathsf{B}}},$$

which holds by Assumption 4.2. We thus find, for $t \in \tilde{J}_k$,

$$\mu_k \int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{\cosh(\sigma\mu_k\delta)} \chi(\frac{1}{c_2} p \operatorname{Re} \hat{f}_k \overline{\hat{u}_k} - \frac{c_1\gamma}{2c_2} p\mu_k^2 |\hat{u}_k|^2)\, dx_n \leq \int_0^\delta \frac{\mathsf{D} e^{-2(\frac{\sigma\mu_k}{2} x_n - sa\eta_{|S})}}{2\gamma \inf_{t\in[0,T]} (c_1 c_2)} |\hat{f}_k|^2\, dx_n,$$

and proceeding as in 2.(a) we find

$$\mu_k \int_{\tilde{J}_k} \int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{(c_{2-} + c_{2+})\cosh(\sigma\mu_k\delta)} c_{2+}\chi\Big(\frac{1}{c_2} p \operatorname{Re} \hat{f}_k \overline{\hat{u}_k} - \frac{c_1\gamma}{2c_2} p\mu_k^2 |\hat{u}_k|^2\Big)\, dx_n\, dt \lesssim \|e^{sa\eta} \hat{f}_k\|^2_{L^2(q_{T,\delta})}.$$

(c) **Term $\chi p \frac{\partial_{x_n} c_2}{c_2} \operatorname{Re} \hat{u}_k \partial_{x_n} \overline{\hat{u}_k}$.**
With the Young inequality we find $\mu_k \chi p \operatorname{Re} \hat{u}_k \partial_{x_n} \overline{\hat{u}_k} \leq \frac{1}{2} p |\partial_{x_n} \hat{u}_k|^2 + \mu_k^2 \frac{1}{2} p |\hat{u}_k|^2$. With (4.22), arguing as above we obtain

$$\begin{aligned}
&\mu_k \int_{\tilde{J}_k} \int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{(c_{2+} + c_{2-})\cosh(\sigma\mu_k\delta)} c_{2+}\chi p \frac{\partial_{x_n} c_2}{c_2} \operatorname{Re} \hat{u}_k \partial_{x_n} \overline{\hat{u}_k} dx_n dt \\
&\qquad \lesssim s\lambda \|(a\varphi)^{\frac{1}{2}} e^{sa\eta} \partial_{x_n} \hat{u}_k\|^2_{L^2(q_{T,\delta})} + s\lambda \|(a\varphi)^{\frac{1}{2}} \mu_k \hat{v}_k\|^2_{L^2(q_{T,\delta})} \\
&\qquad \lesssim s\lambda \|(a\varphi)^{\frac{1}{2}} \partial_{x_n} \hat{v}_k\|^2_{L^2(q_{T,\delta})} + s^3\lambda^3 \|(a\varphi)^{\frac{3}{2}} \hat{v}_k\|^2_{L^2(q_{T,\delta})} + s\lambda \|(a\varphi)^{\frac{1}{2}} \mu_k \hat{v}_k\|^2_{L^2(q_{T,\delta})}.
\end{aligned}$$

(d) **Term $\frac{\chi'}{2c_2} p |\hat{u}_k|^2$.**
As we have $\|\chi'\|_\infty \leq C/t_k$ we get

$$\|\chi'\|_\infty \lesssim T a(t_k) \lesssim \frac{T\Phi(t_k; s, \lambda)}{s\lambda\varphi_{|S}} \sqrt{\frac{\mathsf{D}}{\mathsf{B}}} \lesssim \frac{T\mu_k}{s\lambda\varphi_{|S}} \sqrt{\frac{\mathsf{D}}{\mathsf{B}}}.$$

We thus find

$$\mu_k \int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{\cosh(\sigma\mu_k\delta)} \frac{\chi'}{2c_2} p |\hat{u}_k|^2\, dx_n \lesssim \mu_k^2 T a(t) \sqrt{\frac{\mathsf{D}^3}{\mathsf{B}}} \int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{\cosh(\sigma\mu_k\delta)} e^{2sa\eta_{|S}} |\hat{u}_k|^2\, dx_n.$$

Arguing as above with (4.22) we obtain

$$\begin{aligned}
\mu_k \int_{\tilde{J}_k} \int_0^\delta \frac{\sinh(\sigma\mu_k(\delta - x_n))}{(c_{2+} + c_{2-})\cosh(\sigma\mu_k\delta)} c_{2+} \frac{\chi'}{2\sigma c_2} p |\hat{u}_k|^2\, dx_n\, dt &\lesssim T \|a^{\frac{1}{2}} \mu_k \hat{v}_k\|^2_{L^2(q_{T,\delta})} \\
&\lesssim s\lambda \|(a\varphi)^{\frac{1}{2}} \mu_k \hat{v}_k\|^2_{L^2(q_{T,\delta})},
\end{aligned}$$

if $s \geq s_0 T$, with $s_0 > 0$, and $\lambda \geq \lambda_0 > 0$.

Collecting all the estimates we have obtained we conclude the proof of the Proposition 4.5. ■

End of the proof of Theorem 1.5.

With Proposition 4.1, estimate (4.4), Lemma 4.4 and Proposition 4.5, for λ and $s/(T+T^2)$ sufficiently large, we obtain, for all $k \in \mathbb{N}^*$,

$$\begin{aligned} & s^{-1}\|(a\varphi)^{-\frac{1}{2}}\partial_t \hat{v}_k\|^2_{L^2(q_{T,\delta})} + s\lambda^2\Big(\|(a\varphi)^{\frac{1}{2}}\partial_{x_n}\hat{v}_k\|^2_{L^2(q_{T,\delta})} + \|(a\varphi)^{\frac{1}{2}}\mu_k\hat{v}_k\|^2_{L^2(q_{T,\delta})}\Big) + s^3\lambda^4\|(a\varphi)^{\frac{3}{2}}\hat{v}_k\|^2_{L^2(q_{T,\delta})} \\ & \quad + s\lambda\Big(|(a\varphi_{|S})^{\frac{1}{2}}\partial_{x_n}\hat{v}_k|^2_{L^2((0,T))} + |(a\varphi_{|S})^{\frac{1}{2}}\mu_k\hat{v}_k|^2_{L^2((0,T))}\Big) + s^3\lambda^3|(a\varphi_{|S})^{\frac{3}{2}}\hat{v}_k|^2_{L^2((0,T))} \lesssim \|e^{sa\eta}\hat{f}_k\|^2_{L^2(q_{T,\delta})}. \end{aligned} \tag{4.25}$$

Summing over k, using (2.2) we obtain

$$\begin{aligned} & s^{-1}\|(a\varphi)^{-\frac{1}{2}}\partial_t v\|^2_{L^2(Q_T)} + s\lambda^2\Big(\|(a\varphi)^{\frac{1}{2}}\partial_{x_n}v\|^2_{L^2(Q_T)} + \|(a\varphi)^{\frac{1}{2}}\mu_k v\|^2_{L^2(Q_T)}\Big) + s^3\lambda^4\|(a\varphi)^{\frac{3}{2}}v\|^2_{L^2(Q_T)} \\ & \quad + s\lambda\Big(|(a\varphi_{|S})^{\frac{1}{2}}\partial_{x_n}v|^2_{L^2(S_T)} + |(a\varphi_{|S})^{\frac{1}{2}}\mu_k v|^2_{L^2(S_T)}\Big) + s^3\lambda^3|(a\varphi_{|S})^{\frac{3}{2}}v|^2_{L^2(S_T)} \lesssim \|e^{sa\eta}f\|^2_{L^2(Q_T)}. \end{aligned} \tag{4.26}$$

The remainder of the proof of the Carleman estimate is now classical (see e.g. [24]).

Remark 4.8. It is important to note that Proposition 4.5 is not a trace result, otherwise a stronger Sobolev norm would appear on the r.h.s. of (4.9). The L^2-norm of the trace of the tangential derivative is estimated by an $L^2((0,T); H^1(\Omega))$-norm, but this is valid only for solutions of $Pu = f$. This result appears to us as an expression of the parabolic regularization effect.

Observe that the estimate of Proposition 4.5 is also valid in the case where c_1 and c_2 are smooth *if* the weight function β is chosen with a discontinuous derivative across S according to Assumptions 2.1 and 4.2.

A Proof of some intermediate results

A.1 Proof of Proposition 3.1

For later use of this proof in Section 4 we consider the case $c_1 \neq c_2$ here. The inequality we prove is uniform w.r.t. k. We shall thus remove the Fourier notation $\hat{u}_k$ and simply write $(-\partial_{x_n}c_2\partial_{x_n} + c_1\mu^2)u = f$. We introduce $v = e^{s\varphi}u$ and $g = e^{s\varphi}f$ and we obtain

$$(-\partial_{x_n}c_2\partial_{x_n} - c_2(s\varphi')^2 + c_1\mu^2 + 2sc_2\varphi'\partial_{x_n} + s\partial_{x_n}(c_2\varphi'))v = g,$$

which, following [16], we write $M_1v + M_2v = \tilde{g}$, with

$$\begin{aligned} & M_1 = -\partial_{x_n}c_2\partial_{x_n} - c_2(s\varphi')^2 + c_1\mu^2, \quad M_2 = 2sc_2\varphi'\partial_{x_n} + spc_2\varphi'', \\ & \tilde{g} = g + (p-1)sc_2\varphi''v - s(\partial_{x_n}c_2)\varphi'v, \quad 1 < p < 3. \end{aligned}$$

The introduction of the parameter p is for instance explained in [24]. Following the classical method to prove Carleman estimates we compute

$$\|\tilde{g}\|^2_{L^2(\mathbb{R}^+)} = \|M_1v\|^2_{L^2(\mathbb{R}^+)} + \|M_2v\|^2_{L^2(\mathbb{R}^+)} + 2\,\mathrm{Re}(M_1v, M_2v)_{L^2(\mathbb{R}^+)}, \tag{A.1}$$

considering only the region $\{x_n > 0\}$ for now. We focus on the computation of the third term which we write as sum of 4 terms I_{ij}, $1 \le i \le 2$, $1 \le j \le 2$, where I_{ij} is the inner product of the i^{th} term in the expression of $M_1 v$ and the j^{th} term in the expression of $M_2 v$ above.

Term I_{11}. With an integration by parts we have

$$\begin{aligned} I_{11} &= -2\,\mathrm{Re} \int_{x_n>0} s\varphi'(\partial_{x_n} c_2 \partial_{x_n} v) c_2 \overline{\partial_{x_n} v}\, dx_n = -\int_{x_n>0} s\varphi' \partial_{x_n} |c_2 \partial_{x_n} v|^2\, dx_n \\ &= s\varphi' |c_2 \partial_{x_n} v|^2_{|x_n=0^+} + \int_{x_n>0} s\varphi'' |c_2 \partial_{x_n} v|^2\, dx_n. \end{aligned}$$

Term I_{21}. Similarly we find

$$\begin{aligned} I_{21} &= \mathrm{Re} \int_{x_n>0} \big(- c_2 (s\varphi')^3 + s c_1 \mu^2 \varphi' \big)\, c_2 \partial_{x_n} |v|^2\, dx_n \\ &= (c_2 (s\varphi')^3 - s c_1 \mu^2 \varphi') c_2 |v|^2_{|x_n=0^+} + \int_{x_n>0} c_2 (3 s^3 c_2 (\varphi')^2 \varphi'' - s c_1 \mu^2 \varphi'') |v|^2\, dx_n \\ &\quad + \int_{x_n>0} (2 c_2 \partial_{x_n} c_2 (s\varphi')^3 - (c_1 \partial_{x_n} c_2 + c_2 \partial_{x_n} c_1) s \mu^2 \varphi') |v|^2\, dx_n. \end{aligned}$$

Term I_{12}. We have

$$\begin{aligned} I_{12} &= -sp\,\mathrm{Re} \int_{x_n>0} (\partial_{x_n} c_2 \partial_{x_n} v) c_2 \varphi'' \overline{v}\, dx_n \\ &= sp \int_{x_n>0} \varphi'' |c_2 \partial_{x_n} v|^2\, dx_n + sp\varphi''\, \mathrm{Re}(c_2 \partial_{x_n} v) c_2 \overline{v}_{|x_n=0^+} + sp\,\mathrm{Re} \int_{x_n>0} \partial_{x_n} (c_2 \varphi'') (c_2 \partial_{x_n} v) \overline{v}\, dx_n. \end{aligned}$$

Term I_{22}. We directly find $I_{22} = sp \int_{x_n>0} c_2 \big(- c_2 (s\varphi')^2 + c_1 \mu^2 \big) \varphi'' |v|^2\, dx_n$.

Collecting together the different terms we have obtained we find

$$\frac{1}{2} \|\tilde{g}\|^2_{L^2(\mathbb{R}^+)} \ge \int_{x_n>0} \alpha_0 |v|^2\, dx_n + \int_{x_n>0} \alpha_1 |c_2 \partial_{x_n} v|^2\, dx_n + \gamma_0 |v|^2_{|x_n=0^+} + \gamma_1 |c_2 \partial_{x_n} v|^2_{|x_n=0^+} + X + Y,$$

with

$$\begin{aligned} \alpha_0 &= s(p-1) c_1 c_2 \mu^2 \varphi'' + (3-p) s^3 (c_2 \varphi')^2 \varphi'', \qquad \alpha_1 = s(p+1)\varphi'', \\ \gamma_0 &= c_2^2 (s\varphi')^3_{|x_n=0^+} - c_1 c_2 s \mu^2 \varphi'_{|x_n=0^+}, \quad \gamma_1 = s\varphi', \\ X &= sp\,\mathrm{Re} \int_{x_n>0} \partial_{x_n} (c_2 \varphi'') (c_2 \partial_{x_n} v) \overline{v}\, dx_n \\ &\quad + \int_{x_n>0} (2 c_2 \partial_{x_n} c_2 (s\varphi')^3 - (c_1 \partial_{x_n} c_2 + c_2 \partial_{x_n} c_1) \mu^2 s \varphi') |v|^2\, dx_n, \\ Y &= spc^2 \varphi''\, \mathrm{Re}(\partial_{x_n} v) \overline{v}_{|x_n=0^+}. \end{aligned}$$

Because of the form of φ, a direct computation shows that

$$\alpha_0 \gtrsim s\lambda^2 \mu^2 \varphi + s^3 \lambda^4 \varphi^3, \quad \alpha_1 \gtrsim C s \lambda^2 \varphi,$$

for λ chosen sufficiently large. Recalling that β is affine in the region we consider we find

$$\begin{aligned} X &= sp\,\mathrm{Re} \int_{x_n>0} (c_2^2 \lambda^3 \beta'^3 + c_2 (\partial_{x_n} c_2) \lambda^2 \beta'^2) \varphi (\partial_{x_n} v) \overline{v}\, dx_n \\ &\quad + \int_{x_n>0} (2 c_2 \partial_{x_n} c_2 (s\lambda\beta'\varphi)^3 - (c_1 \partial_{x_n} c_2 + c_2 \partial_{x_n} c_1) \mu^2 s \lambda \beta' \varphi) |v|^2\, dx_n, \end{aligned}$$

and

$$\|\tilde{g}\|^2_{L^2(\mathbb{R}^+)} \lesssim \|g\|^2_{L^2(\mathbb{R}^+)} + s^2(\lambda^4 + \lambda^2) \int_{x_n>0} \varphi^2|v|^2.$$

Choosing s and λ sufficiently large, with the Young inequality, we obtain

$$\begin{aligned} C\|g\|^2_{L^2(\mathbb{R}^+)} \geq C' \int_{x_n>0} (s\lambda^2\mu^2\varphi + s^3\lambda^4\varphi^3)|v|^2 \, dx_n + C' \int_{x_n>0} s\lambda^2\varphi|\partial_{x_n} v|^2 \, dx_n \\ + \gamma_0|v|^2_{|x_n=0^+} + \gamma_1|c_2\partial_{x_n} v|^2_{|x_n=0^+} + Y. \end{aligned} \tag{A.2}$$

The same type of estimate can be obtained in the region $\{x_n < 0\}$ with opposite signs for the trace terms. The sum of (A.2) from both sides yields the result. ∎

A.2 Proof of Lemma 3.4

Here we drop the $\hat{v}_k$ notation and simply write v. It follows that

$$c\partial_{x_n} v = e^{s\varphi}(c\partial_{x_n} u + cs(\partial_{x_n}\varphi)u) = e^{s\varphi}(c\partial_{x_n} u + c\beta'(s\lambda\varphi u)).$$

We set $a = c\partial_{x_n} u$ and $b = s\lambda\varphi u$. We then have

$$[|c\partial_{x_n} v|^2\beta']_S = e^{2s\varphi}\Big([\beta']_S|a|^2 + [c^2(\beta')^3]_S|b|^2 + 2[c(\beta')^2] \operatorname{Re} a\overline{b}\Big).$$

We thus obtain

$$\mathcal{B}(v) = s\lambda\varphi_{|S}\, e^{2s\varphi}(Aw, w),$$

with $w = (a, b)^t$ and where A is the following symmetric matrix

$$A = \begin{pmatrix} [\beta']_S & [c(\beta')^2]_S \\ [c(\beta')^2] & 2[c^2(\beta')^3]_S \end{pmatrix} = \beta'_- \begin{pmatrix} (L-1) & c_+\beta'_-(L^2 - K) \\ c_+\beta'_-(L^2 - K) & 2(c_+\beta'_-)^2(L^3 - K^2) \end{pmatrix}.$$

We then see that

$$(Aw, w) = \beta'_-(L-1)\Big|a + c_+\beta'_- \frac{L^2 - K}{L-1} b\Big|^2 + \beta'_-\Big(2(c_+\beta'_-)^2(L^3 - K^2) - (c_+\beta'_-)^2 \frac{(L^2 - K)^2}{L-1}\Big)|b|^2.$$

which gives the result. ∎

A.3 Traces of the solution

We consider the following ODEs

$$v'' - \mu^2 v = F, \quad s \in (-\delta, 0) \cup (0, \delta), \tag{A.3}$$

$$v(-\delta) = v(\delta) = 0, \quad v_{|s=0^-} = v_{|s=0^+}, \quad cv'_{|s=0^-} = cv'_{|s=0^+}. \tag{A.4}$$

Here $\mu > 0$. The solutions of (A.3) can be written as

$$v(s) = A_\pm \cosh(\mu s) + B_\pm \sinh(\mu s) + \mu^{-1} \int_0^s \sinh(\mu(s - \sigma))F(\sigma) \, d\sigma, \quad s \in (-\delta, 0) \cup (0, \delta).$$

We then have $A_{\pm} = v_{|s=0^{\pm}}$, $\mu B_{\pm} = v_{|s=0^{\pm}}$ and

$$v(\pm\delta) = \mu^{-1}\cosh(\mu\delta)\left(\mu A_{\pm} + \mu B_{\pm}\tanh(\pm\mu\delta) + \int_0^{\pm\delta}\frac{\sinh(\mu(\pm\delta-\sigma))}{\cosh(\mu\delta)}F(\sigma)\,d\sigma\right).$$

The boundary conditions (A.4) then yield

$$\begin{pmatrix}\mu & \tanh(\mu\delta)\\ \mu & -\frac{c_+}{c_-}\tanh(\mu\delta)\end{pmatrix}\begin{pmatrix}v_+(0)\\ v_+'(0)\end{pmatrix} = \begin{pmatrix}-\int_0^{\delta}\frac{\sinh(\mu(\delta-\sigma))}{\cosh(\mu\delta)}F(\sigma)\,d\sigma\\ -\int_0^{-\delta}\frac{\sinh(\mu(-\delta-\sigma))}{\cosh(\mu\delta)}F(\sigma)\,d\sigma\end{pmatrix}.$$

We observe that the determinant of this system,

$$D = -c_-^{-1}\tanh(\mu\delta)\mu(c_+ + c_-),$$

is non zero as $\mu > 0$. It thus follows that

$$v_-(0) = v_+(0) = -\frac{c_+}{\mu}\int_0^{\delta}\frac{\sinh(\mu(\delta-\sigma))}{(c_+ + c_-)\cosh(\mu\delta)}F(\sigma)\,d\sigma - \frac{c_-}{\mu}\int_0^{-\delta}\frac{\sinh(\mu(-\delta-\sigma))}{(c_+ + c_-)\cosh(\mu\delta)}F(\sigma)\,d\sigma.$$

A.4 Proof of Lemma 3.6

We first consider the case $K = 1$. Then $L = 2$ and $P(K, L) = -(1+\varepsilon) + 5(1-\varepsilon)^2$. The result is clear for ε sufficiently small.

We now consider the case $0 < K < 1$. Then $L > 1$; we have $L - K^2 > 0$ and thus

$$P(K, L) = -(L-K^2)(L-1) + \frac{(1-\varepsilon)^2}{1+\varepsilon}\left(2(L^3-K^2)(L-1) - (L^2-K)^2\right).$$

For convenience we write $(1-\varepsilon)^2/(1+\varepsilon) = 1-\alpha$ with $0 < \alpha < 1$. We then find

$$Q(K) = P(K, (r+1) - rK) = -(K-1)^2 S(K), \quad S(K) = aK^2 + bK + c,$$

with

$$a = -(1-\alpha)r^4 < 0, \quad b = 2(1-\alpha)(r^4 + r^3 - r^2) - (1-2\alpha)r,$$
$$c = -(1-\alpha)(r^4 + 2r^3 - 1) + r^2 + (3-2\alpha)r.$$

As S is a concave quadratic polynomial it suffices to prove that $S(1) \le 0$ and $S'(1) \ge 0$. We find

$$S(1) = (2\alpha-1)r^2 + 2r + 1 - \alpha, \quad S'(1) = r(2(1-al)r^2 - 2(1-\alpha)r - (1-2\alpha)).$$

We see that $S(1) < 0$ and $S'(1) > 0$ if $\alpha = 0$ and $r = 3$. It thus remains true for α sufficiently small. ■

A.5 Proof of Proposition 4.1

The inequality we prove is uniform w.r.t. k. We shall thus remove the Fourier notation $\hat{u}_k$ and simply write $(\partial_t - \partial_{x_n} c_2 \partial_{x_n} + c_1\mu^2)u = f$. We introduce $v = e^{sa\eta}u$ and $g = e^{sa\eta}f$ and we obtain

$$\Big(\partial_t - \partial_{x_n} c_2 \partial_{x_n} - c_2(sa\eta')^2 + c_1\mu^2 + 2sc_2 a\eta'\partial_{x_n} + sa\partial_{x_n}(c_2\eta') - sa'\eta\Big)v = g,$$

which we write $M_1 v + M_2 v = \tilde{g}$, with

$$\begin{aligned} &M_1 = -\partial_{x_n} c_2 \partial_{x_n} + [-c_2(sa\eta')^2 + c_1\mu^2] - sa'\eta, \quad M_2 = 2sc_2 a\eta'\partial_{x_n} + spc_2 a\eta'' + \partial_t, \\ &\tilde{g} = g + (p-1)sc_2 a\eta'' v - s(\partial_{x_n} c_2)a\eta' v, \quad 1 < p < 3. \end{aligned} \tag{A.5}$$

In preparation for what follows we observe that

$$1 \lesssim T^2 a, \quad |a'| \lesssim Ta^2, \quad |a''| \lesssim T^2 a^3, \quad |\eta| \lesssim \varphi^2.$$

We have

$$\|\tilde{g}\|^2_{L^2((0,T)\times\mathbb{R}^+)} \lesssim \|g\|^2_{L^2((0,T)\times\mathbb{R}^+)} + s^2(\lambda^4 + \lambda^2)T^2 \|a^{\frac{3}{2}}\varphi^{\frac{1}{2}} v\|^2_{L^2((0,T)\times\mathbb{R}^+)}.$$

We compute

$$\|\tilde{g}\|^2_{L^2((0,T)\times\mathbb{R}^+)} = \|M_1 v\|^2_{L^2((0,T)\times\mathbb{R}^+)} + \|M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^+)} + 2\,\mathrm{Re}(M_1 v, M_2 v)_{L^2((0,T)\times\mathbb{R}^+)}, \tag{A.6}$$

considering only the region $\{x_n > 0\}$ for now. For the computation of the last term in (A.6), we set I_{ij}, $1 \le i \le 3$, $1 \le j \le 3$, where I_{ij} is the inner product of the i^{th} term in the expression of $M_1 v$ and the j^{th} term in the expression of $M_2 v$ above. For the computations of I_{11}, I_{12}, I_{21} and I_{22} we refer to the computations performed in Appendix A.1 (simply replacing φ by $a\varphi$ and integrating in time).

Term I_{13}. By integration by parts we find

$$\begin{aligned} I_{13} &= \mathrm{Re} \int_0^T \int_{x_n>0} -(\partial_{x_n} c_2 \partial_{x_n} v)\partial_t \overline{v}\, dx_n dt \\ &= \frac{1}{2}\int_0^T \int_{x_n>0} c_2 \partial_t |\partial_{x_n} v|^2\, dx_n dt + \mathrm{Re}\int_0^T ((c_2\partial_{x_n} v)\partial_t \overline{v})_{|x_n=0^+}\, dt \\ &= -\frac{1}{2}\int_0^T \int_{x_n>0} (\partial_t c_2)|\partial_{x_n} v|^2\, dx_n dt + \mathrm{Re}\int_0^T ((c_2\partial_{x_n} v)\partial_t \overline{v})_{|x_n=0^+}\, dt. \end{aligned}$$

We have

$$\left|\frac{1}{2}\int_0^T \int_{x_n>0} (\partial_t c_2)|\partial_{x_n} v|^2\, dx_n dt\right| \lesssim T^2 \int_0^T \int_{x_n>0} a|\partial_{x_n} v|^2\, dx_n dt.$$

Term I_{23}. By integration by parts we have

$$\begin{aligned} I_{23} = \frac{1}{2}\int_0^T \int_{x_n>0} (-c_2(sa\eta')^2 + c_1\mu^2)\partial_t |v|^2\, dx_n dt &= s^2 \int_0^T \int_{x_n>0} c_2 a a' \eta'^2 |v|^2\, dx_n dt \\ &\quad + \frac{1}{2}\int_0^T \int_{x_n>0} [(\partial_t c_2)(sa\eta')^2 - (\partial_t c_1)\mu^2]|v|^2\, dx_n dt \end{aligned}$$

We thus find

$$|I_{23}| \lesssim (T+T^2)s^2\lambda^2 \int_0^T \int_{x_n>0} a^3\varphi^2 |v|^2 \, dx_n dt + T^2 \int_0^T \int_{x_n>0} a\mu^2 |v|^2 \, dx_n dt.$$

Term I_{33}. By integration by parts we find

$$I_{33} = -\frac{s}{2} \int_0^T \int_{x_n>0} a'\eta \partial_t |v|^2 \, dx_n dt = \frac{s}{2} \int_0^T \int_{x_n>0} a''\eta |v|^2 \, dx_n dt.$$

This yields

$$|I_{33}| \lesssim T^2 s \int_0^T \int_{x_n>0} a^3\varphi^2 |v|^2 \, dx_n dt.$$

Terms I_{31} and I_{32}. We have

$$I_{31} = -s^2 \int_0^T \int_{x_n>0} c_2 a a' \eta\eta' \partial_{x_n} |v|^2 \, dx_n dt = s^2 \int_0^T \int_{x_n>0} a a' \partial_{x_n}(c_2\eta\eta') |v|^2 \, dx_n dt.$$

We find directly $I_{32} = -ps^2 \int_0^T \int_{x_n>0} c_2 a a' \eta\eta'' |v|^2 \, dx_n dt$. We then obtain

$$|I_{31}| + |I_{32}| \lesssim T s^2(\lambda+\lambda^2) \int_0^T \int_{x_n>0} a^3\varphi^3 |v|^2 \, dx_n dt.$$

With the computations of Appendix A.1 we find, for λ and $sa \gtrsim s/T^2$ sufficiently large,

$$\begin{aligned} C\|g\|^2_{L^2((0,T)\times\mathbb{R}^+)} &\geq C' \int_0^T \int_{x_n>0} (s\lambda^2\mu^2 a\varphi + s^3\lambda^4 (a\varphi)^3)|v|^2 \, dx_n dt + C' \int_0^T \int_{x_n>0} s\lambda^2 a\varphi |\partial_{x_n} v|^2 \, dx_n dt \\ &+ \frac{1}{2}\|M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^+)} + \int_0^T (\gamma_0 |v|^2_{|x_n=0^+} + \gamma_1 |c\partial_{x_n} v|^2_{|x_n=0^+}) \, dt + X + Y + I_{13} + I_{23} + I_{33} + I_{31} + I_{32}, \end{aligned}$$

with

$$\begin{aligned} \gamma_0 &= c_2^2 (sa\varphi')^3_{|x_n=0^+} - c_1 c_2 s\mu^2 a\varphi'_{|x_n=0^+}, \quad \gamma_1 = sa\varphi', \\ X &= sp \operatorname{Re} \int_0^T \int_{x_n>0} a\partial_{x_n}(c_2\varphi'')(c_2\partial_{x_n} v)\bar{v} \, dx_n dt \\ &\quad + \int_0^T \int_{x_n>0} [2c_2(\partial_{x_n} c_2)(sa\varphi')^3 - \partial_{x_n}(c_1 c_2)\mu^2 sa\varphi'] |v|^2 \, dx_n dt, \\ Y &= sp \int_0^T c_2^2 a\varphi'' \operatorname{Re}(\partial_{x_n} v)\bar{v}_{|x_n=0^+} \, dt. \end{aligned}$$

For λ and $s/(T+T^2)$ sufficiently large, the estimations we found above yield,

$$\begin{aligned} C\|g\|^2_{L^2((0,T)\times\mathbb{R}^+)} &\geq C' \int_0^T \int_{x_n>0} (s\lambda^2\mu^2 a\varphi + s^3\lambda^4 (a\varphi)^3)|v|^2 \, dx_n dt + C' \int_0^T \int_{x_n>0} s\lambda^2 a\varphi |\partial_{x_n} v|^2 \, dx_n dt \\ &\quad + \frac{1}{2}\|M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^+)} + \int_0^T (\gamma_0 |v|^2_{|x_n=0^+} + \gamma_1 |c_2\partial_{x_n} v|^2_{|x_n=0^+}) \, dt + \check{Y}, \quad \text{(A.7)} \end{aligned}$$

with

$$\check{Y} = sp\int_0^T c_2^2 a\varphi'' \operatorname{Re}(\partial_{x_n} v)\overline{v}_{|x_n=0^+}\, dt + \operatorname{Re}\int_0^T ((c_2\partial_{x_n} v)\partial_t \overline{v}_{|x_n=0^+}\, dt.$$

The same type of estimate can be obtained in the region $\{x_n < 0\}$ with opposite signs for the trace terms. The sum of (A.7) from both sides yields

$$\begin{aligned} C\Big(s\lambda^2 \|(a\varphi)^{\frac{1}{2}}\partial_{x_n} v\|^2_{L^2(Q_{T,\delta})} + s\lambda^2\|(a\varphi)^{\frac{1}{2}}\mu v\|^2_{L^2(Q_{T,\delta})} + s^3\lambda^4\|(a\varphi)^{\frac{3}{2}} v\|^2_{L^2(Q_{T,\delta})}\Big) \\ + \frac{1}{2}\|M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^*)} + s\lambda\int_0^T a\varphi_{|S}\left([c_2^2\beta'|\partial_{x_n} v|^2]_S + |s\lambda a\varphi v_{|S_T}|^2[c_2^2\beta'^3]_S\right) dt + \tilde{Y} \\ \leq C\|e^{sa\eta} f\|^2_{L^2(Q_{T,\delta})} + s\lambda\int_0^T a\varphi_{|S}\, |\mu v_{|S_T}|^2[c_1 c_2\beta']_S\, dt, \quad \text{(A.8)} \end{aligned}$$

with $\tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2$, where

$$\tilde{Y}_1 = sp\lambda^2\int_0^T a\varphi_{|S} \operatorname{Re}\,[c_2^2\beta'^2\partial_{x_n} v]_S\, \overline{v}_{|S_T}\, dt,$$

and

$$\tilde{Y}_2 = \operatorname{Re}\int_0^T [c_2\partial_{x_n} v]_S\, \partial_t \overline{v}_{|S_T}\, dt.$$

We have

$$\begin{aligned} |\tilde{Y}_1| &\lesssim s^{\frac{1}{2}}\lambda\int_0^T a^{\frac{1}{2}}\varphi_{|S}\,(|\partial_{x_n} v|^2_{|S_T^-} + |\partial_{x_n} v|^2_{|S_T^+})\, dt + s^{\frac{3}{2}}\lambda^3\int_0^T a^{\frac{3}{2}}\varphi_{|S}\,|v|^2_{|S_T}\, dt \\ &\lesssim s^{\frac{1}{2}}\lambda T\int_0^T a\varphi_{|S}\,(|\partial_{x_n} v|^2_{|S_T^-} + |\partial_{x_n} v|^2_{|S_T^+})\, dt + s^{\frac{3}{2}}\lambda^3 T^3\int_0^T a^3\varphi_{|S}\,|v|^2_{|S_T}\, dt. \end{aligned}$$

As $u = e^{-sa\eta}v$ we have $c\partial_{x_n} u = c_2(\partial_{x_n} v - sa(\eta')v)e^{-sa\eta}$ and thus

$$[c_2\partial_{x_n} v]_S = sa[c_2\eta']_S\, v_{|S_T} = s\lambda a[c_2\beta']_S\,(v\varphi)_{|S_T}.$$

By integration by parts, we thus have

$$\tilde{Y}_2 = \frac{1}{2}s\lambda\int_0^T a[c_2\partial_{x_n}\beta]_S\,\varphi_{|S}\,\partial_t|v|^2_{|S_T}\, dt = -\frac{1}{2}s\lambda\int_0^T [\partial_t(ac_2)\partial_{x_n}\beta]_S\,\varphi_{|S}\,|v|^2_{|S_T}\, dt.$$

We thus obtain

$$|\tilde{Y}_2| \lesssim s(T^3 + T^4)\lambda\int_0^T a^3\varphi_{|S}\,|v|^2_{|S_T}\, dt. \quad \text{(A.9)}$$

Finally, from the form of M_2 in (A.5), we have

$$\begin{aligned} \|(sa\varphi)^{-\frac{1}{2}}\partial_t v\|^2_{L^2((0,T)\times\mathbb{R}^*)} &\lesssim \|(sa\varphi)^{-\frac{1}{2}} M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^*)} + \|(sa\varphi)^{\frac{1}{2}}\lambda\partial_{x_n} v\|^2_{L^2(Q_{T,\delta})} + \|(sa\varphi)^{\frac{1}{2}}\lambda^2 v\|^2_{L^2(Q_{T,\delta})} \\ &\lesssim \|M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^*)} + s\lambda^2\|(a\varphi)^{\frac{1}{2}}\partial_{x_n} v\|^2_{L^2(Q_{T,\delta})} + s\lambda^4\|(a\varphi)^{\frac{1}{2}} v\|^2_{L^2(Q_{T,\delta})} \\ &\lesssim \|M_2 v\|^2_{L^2((0,T)\times\mathbb{R}^*)} + s\lambda^2\|(a\varphi)^{\frac{1}{2}}\partial_{x_n} v\|^2_{L^2(Q_{T,\delta})} + s^3\lambda^4\|(a\varphi)^{\frac{3}{2}} v\|^2_{L^2(Q_{T,\delta})}, \end{aligned}$$

as here $sa \gtrsim s/T^2 \geq s_0$, for some $s_0 > 0$ and $\varphi \geq 1$. ■

A.6 Proof of Lemma 4.7

Computing $p'(t) = s\lambda e^{2sa(t)\eta_{|S}} a'(t)\varphi_{|S}[1 + 2s\eta_{|S} a(t)]\mathsf{D}$, we have

$$\frac{p'}{p}(t) = \frac{a'}{a}(t)[1 + 2sa(t)\eta_{|S}],$$

If $\frac{T}{2} \leq t < T - \frac{t_k}{2}$, because of the form of η in (4.1) we find $\frac{p'}{p} \leq 0$, for $sa(t) \gtrsim s/T^2$ and λ both sufficiently large. This implies that inequality (4.18) holds for these values of t.

We now consider the case $\frac{t_k}{2} < t < \frac{T}{2}$. Note that $\frac{p'}{p}$ is nonnegative, for s/T^2 and λ large, as here $a'(t) < 0$. Setting $\tilde{c}_1 = \inf_{t,x_n} c_1(t, x_n)$, and using the definition of t_k in (4.11), it suffices to prove

$$\frac{p'(t)}{\tilde{c}_1 p(t)} \leq (1 - \gamma)\Phi^2(t_k; s, \lambda), \qquad \frac{t_k}{2} < t < \frac{T}{2}.$$

For all s, λ, we have

$$\frac{p'(t)}{p(t)} = \frac{2t - T}{t(T - t)}[2s\eta_{|S} a(t) + 1] \leq \frac{2t - T}{t(T - t)} 2s\eta_{|S} a(t) \leq -2Ts\eta_{|S} a^2(t).$$

As we have

$$\frac{1}{2} < \frac{\Phi(t_k; s, \lambda)}{\Phi(t_k/2; s, \lambda)} = \frac{a(t_k)}{a(t_k/2)} < \frac{3}{4}, \tag{A.10}$$

it is sufficient to prove

$$-\frac{2T}{\tilde{c}_1} s\eta_{|S} a^2(t) \leq \frac{1}{4}(1 - \gamma)\Phi^2(t_k/2; s, \lambda), \qquad \frac{t_k}{2} < t < \frac{T}{2}. \tag{A.11}$$

As the function Φ decreases on $(0, T/2)$, (A.11) holds if we have

$$-\frac{2T}{\tilde{c}_1} s\eta_{|S} a^2(t) \leq \frac{1}{4}(1 - \gamma)\Phi^2(t; s, \lambda), \qquad \frac{t_k}{2} < t < \frac{T}{2}.$$

With the definition of Φ, this reads

$$-\frac{\eta_{|S}}{\varphi_{|S}^2} \leq \frac{\tilde{c}_1}{32T}(1 - \gamma)\frac{\mathsf{B}}{\mathsf{D}} s\lambda^2$$

or equivalently

$$e^{\lambda(\bar{\beta} - 2\beta_{|S})} - e^{-\lambda\beta_{|S}} \leq \frac{\tilde{c}_1}{32T}(1 - \gamma)\frac{\mathsf{B}}{\mathsf{D}} s\lambda^2.$$

As $\bar{\beta} < 2\beta_{|S}$ by construction of β (see the beginning of Section 4), this will hold for λ and s/T sufficiently large. ■

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